Concepts of condensed matter physics - Exercise #2

Spring 2020

Due date: 11/01/2020

1. **Quantization of inter-layer Hall conductivity –** Consider two parallel two-dimensional conductors (e.g. two layers of graphene separated by a thin insulating buffer). In the presence of a perpendicular magnetic field the generalized conductivity tensor has the following form

$$J_i^a = \sigma_{ij}^{ab} E_j^b$$

where $a = 1, 2$ denotes the layer number and J_i^a and E_i^a are the i'th component of the in-plane current density and electric field. Show that (also) the off-diagonal Hall conductivity σ_{xy}^{12} is quantized following the arguments presented in class. Do so following these steps:

   a. Write the generalized linear response formula for this off diagonal element, $\sigma_{xy}^{12} = \frac{J_x^1}{\epsilon_y^2}$.

   b. Using the appropriate toroidal geometry, write this element in terms of derivatives of the ground state with respect to magnetic fluxes threaded through the torus' holes.

   c. Generalize the argument given in class to prove that σ_{xy}^{12} is quantized. What happens when the two layers are decoupled?

2. **A simple tight-binding model for a 2D topological insulator -** Discuss σ_{xy} for spin-less particles on a square lattice model that has the following Hamiltonian

$$H = \sum_{\boldsymbol{k}} (\psi_s^+(\boldsymbol{k}) \quad \psi_p^+(\boldsymbol{k}))\ \widehat{H}(\boldsymbol{k}) \begin{pmatrix} \psi_s(\boldsymbol{k}) \\ \psi_p(\boldsymbol{k}) \end{pmatrix}, \text{ where}$$

$$\widehat{H}(\boldsymbol{k}) = A\left(\sin k_x \tau_x + \sin k_y \tau_y\right) + \left(m - t\cos k_x - t\cos k_y\right)\tau_z.$$

Here the τ's are Pauli matrices acting in the orbital basis.

   a. Find the corresponding real-space representation of the tight-binding Hamiltonian.

   b. Discuss σ_{xy} as a function of m

c. Plot the pseudo spin configuration as a function of $e = m/t$ for different values of e_a -- choose them wisely.

d. Assume that the crystal exists only for $x < 0$, and that for $x > 0$ there is vacuum. Write the Schrodinger equation for the single particle solutions near the Fermi energy and (assume that $m > 0$ and that e is close to the critical value).

i. What are the boundary conditions at $x = 0$?

ii. What are the conditions for the existence of a gapless solution on the boundary?

iii. What is the decay length of the wave function?

iv. What happens to the solution at the critical value of the parameter e?

e. Now assume that the crystal exists for all x. Consider the situation where for $x < 0$ the parameter e is slightly larger than the critical value, and for $x > 0$ the parameter e is slightly smaller than it. Find the gapless 1D mode residing on the boundary.

f. Can you generalize the model to one that realizes an arbitrary Chern number?

3. **Spin-wave dispersion** (Consult "Interacting Electrons and Quantum Magnetism" by A. Auerbach pages 123 - 126). In this question you are asked to derive the spin-wave dispersion of the two-dimensional Heisenberg model on a square lattice with antiferromagnetic coupling (i.e. $J > 0$). The Hamiltonian of such a model is given by

$$H = J\sum_{\langle ij\rangle}\left[\,S_i^z S_j^z + \frac{1}{2}\left(S_i^+ S_j^- + S_i^- S_j^+\right)\right]$$

where $S^+ = (S^x + S^y)/2$ and $S^- = (S^+)^+$ are the spin raising and lowering operators and $S^{x,y,z}$ are the spin-half operators which are arranged on a square lattice. The $\langle ij\rangle$ brackets denote summation over nearest neighbors.

a. Separate the lattice into two sub-lattices, A and B, such that all the neighbors of an A site are B's and vice versa. Now take $\langle S_j^z\rangle = \eta(j)$ where $\eta(j) = 1$ if $j \in A$ and $\eta(j) = -1$ if $j \in B$. What is the ground state-energy given by this solution?

b. Show that the mean-field solution you have obtained is not an eigen-state of the Hamiltonian, and thus is not the true ground state.

c. Now let us refine the solution by accounting for quantum fluctuations. First apply a rotation of π about the x axis to all spins on sub-lattice B $\boldsymbol{S}_j \rightarrow \tilde{\boldsymbol{S}}_j$ (the idea is that we expand the Hamiltonian around the mean-field solution, where we have assumed that the spins are aligned along the z direction and anti-parallel to all their nearest neighbors). Now let us assume that all spins are fluctuating weakly around $\langle \tilde{S}_i^z\rangle \approx \frac{1}{2}$, such that we may introduce the Holstein-Primakoff bosons

$$S^z = \frac{1}{2} - n_b$$

$$S^+ = \sqrt{1 - n_b}\, b$$

$$S^- = b^+ \sqrt{1 - n_b}$$

where $n_b = b^+ b$. Apply this transformation

d. Diagonalize the Bosonic theory using a Bogoliubov transformation. Plot the spin-wave dispersion schematically. (note that in the limit of weak fluctuations $\langle n_b \rangle \ll 1$).

e. In class you have derived the spin-dispersion of the ferromagnetic model (i.e. $j < 0$). Discuss the difference in the long wave-length dependence?