A 2-dimensional topological superconductor:

Consider the following 2D tight binding model of spinless electrons with hopping on a square lattice and superconducting terms of the form:

$$H = H_t + H_\mu + H_\Delta$$

$$H_t = -\left(\sum_{i,j} t\psi_{i,j}^\dagger \psi_{i+1,j} + t\psi_{i,j}^\dagger \psi_{i,j+1}\right) + h.c.$$

$$H_\mu = -\sum_{i,j} \mu\psi_{i,j}^\dagger \psi_{i,j}$$

$$H_\Delta = \frac{\Delta}{2}\sum_{i,j}\left(\psi_{i,j}^\dagger\psi_{i+1,j}^\dagger - \psi_{i,j}^\dagger\psi_{i-1,j}^\dagger + i\psi_{i,j}^\dagger\psi_{i,j+1}^\dagger - i\psi_{i,j}^\dagger\psi_{i,j-1}^\dagger\right) + h.c.$$

a) Explain why in this case we cannot write on-site superconducting terms.

b) Transforming to k-space $\psi_{i',j}^\dagger = \frac{1}{\sqrt{N}}\sum_{\vec{k}}\Psi^\dagger(k)e^{i\vec{k}\cdot(i',j)}$, write the Hamiltonian in the form

$$H = \frac{1}{2}\sum_k \Psi^\dagger(k)h(k)\Psi(k),$$

with $\boldsymbol{\Psi} = \left(\boldsymbol{\psi}(\boldsymbol{k}), \boldsymbol{\psi}^\dagger(-\boldsymbol{k})\right)^{\boldsymbol{T}}$ and $\boldsymbol{h}(\boldsymbol{k}) = \begin{pmatrix} \boldsymbol{\varepsilon}(\boldsymbol{k}) & \boldsymbol{d}(\boldsymbol{k}) \\ \boldsymbol{d}^*(\boldsymbol{k}) & -\boldsymbol{\varepsilon}(-\boldsymbol{k}) \end{pmatrix}$

Diagonalize it and find the spectrum. Is there a gap to excitations? If so, find the values of $\frac{t}{\mu}$ in which the gap closes.

c) Study the Chern number of the model as a function of $\frac{t}{\mu}$.

d) Find a low energy theory near the critical point with a positive $\frac{t}{\mu}$ To do so, expand the Hamiltonian to first order in momentum (around which point?), and get the form

$$h(k) = \alpha\sigma_x k_y + \beta\sigma_y k_x + m\sigma_z.$$

What are α, β, m? (for consistency, make sure $m = 0$ in the transition).

e) Using the above continuum model in real space ($k \rightarrow -i\partial_x$), examine the situation where the parameter m is spatially dependent:

$$m = \begin{cases} m_0, x > 0 \\ -m_0, x < 0 \end{cases}.$$

Show that the system has a zero energy mode localized near $x = 0$. What is its localization length? What happens to it at the critical point.

$$H = H_t + H_\mu + H_\Delta$$

$$H_t = -t \sum_{ij} \left[\psi^\dagger_{ij} \psi_{i+1,j} + \psi^\dagger_{ij} \psi_{i,j+1} + h.c. \right]$$

$$H_\mu = -\mu \sum_{ij} \psi^\dagger_{ij} \psi_{ij}$$

$$H_\Delta = \Delta \sum_{ij} \left[\psi^\dagger_{ij} \psi^\dagger_{i+1,j} + i \psi^\dagger_{ij} \psi^\dagger_{i,j+1} \right] + h.c.$$

a) fermions are spinless and $\psi^\dagger_{ij} \psi^\dagger_{ij} = 0$!

b)

$$H_t = -\frac{t}{N} \sum_{ij} \sum_{k,q} \left[\psi^\dagger_k \psi_q e^{-iq_x} + \psi^\dagger_k \psi_q e^{-iq_y} \right] e^{i(k-q)\cdot r} =$$

$$= -t \sum_k \psi^\dagger_k \psi_k \left[2\cos(k_x) + 2\cos(k_y) \right]$$

$$H_\mu = -\mu \sum_k \psi_k^{+} \psi_k$$

$$H_\Delta = \frac{\Delta}{2} \sum_{ij} \Big[\psi_{ij}^{+} \psi_{i+1,j}^{+} - \psi_{ij}^{+} \psi_{i-1,j}^{+} + i\, \psi_{ij}^{+} \psi_{i,j+1}^{+} - i\, \psi_{ij}^{+} \psi_{i,j-1}^{+} \Big] + h.c. =$$

$$= \frac{\Delta}{2N} \sum_{ij} \sum_{kq} \psi_k^{+} \psi_q^{+} \Big[e^{iq_x} - e^{-iq_x} + i\, e^{iq_y} - i\, e^{-iq_y} \Big] e^{i(q+k)r} + h.c.$$

$$= -\Delta \sum_k \psi_k^{+} \psi_{-k}^{+} \big[i\sin(k_x) - \sin(k_y) \big] + h.c.$$

define $\Psi = (\psi_k \quad \psi_{-k}^{+})^T$

and:

$$H_t + H_\mu = \sum_k \psi_k^{+} \psi_k \, \xi(k)$$

$\xi(k) = -2t\big[\cos(k_x) + \cos(k_y)\big] - \mu$

$$= \frac{1}{2} \sum \big[\psi^{+} \psi \; \xi(\ldots) - \psi^{+} \psi \; \xi(\ldots) \big]$$

$$= \frac{1}{2}\sum_k \left[\psi_k^+ \psi_k \xi(k) + \psi_k \psi_k^+ \xi(k)\right]$$

$$= \frac{1}{2}\sum_k \left[\psi_k^+ \psi_k \xi(k) - \psi_{-k} \psi_{-k}^+ \underset{\xi(k)}{\underset{\shortparallel}{\xi(-k)}}\right]$$

$$+ \frac{1}{2}\sum_k \xi(k) =$$

$$= \frac{1}{2}\sum_k \Psi^+(k) \begin{pmatrix} \xi(k) & 0 \\ 0 & -\xi(k) \end{pmatrix} \Psi(k)$$

$$+ \frac{1}{2}\sum_k \xi(k)$$

$$H_\Delta = \frac{1}{2}\sum_k \Psi^+(k) \begin{pmatrix} 0 & d(k) \\ d^*(k) & 0 \end{pmatrix} \Psi(k)$$

$$d(k) = 2\Delta\left[\sin(k_y) - i\sin(k_x)\right]$$

$$\Longrightarrow H = \frac{1}{2}\sum_k \Psi^+(k)\, h_{BdG}(k)\, \Psi(k)$$

$$+ \frac{1}{2}\sum_k \xi(k)$$

$$n_B d_G(k) = d(k)\cdot\sigma$$

$$d_z(k) = -2t\cos(k_x) - 2t\cos(k_y) - \mu$$

$$d_x(k) = 2\Delta\sin(k_y)$$

$$d_y(k) = 2\Delta\sin(k_x)$$

$$\Rightarrow E = \pm|\vec{d}(k)| = \pm\sqrt{\varepsilon^2(k) + 4\Delta^2[\sin^2(k_x) + \sin^2(k_y)]} =$$

$$\pm\sqrt{[2t\cos(k_x) + 2t\cos(k_y) + \mu]^2 + 4\Delta^2\sin^2(k_x) + 4\Delta^2\sin^2(k_y)}$$

$$E \overset{?}{=} 0 \Rightarrow k_x = 0, \pi$$

$$k_y = 0, \pi$$

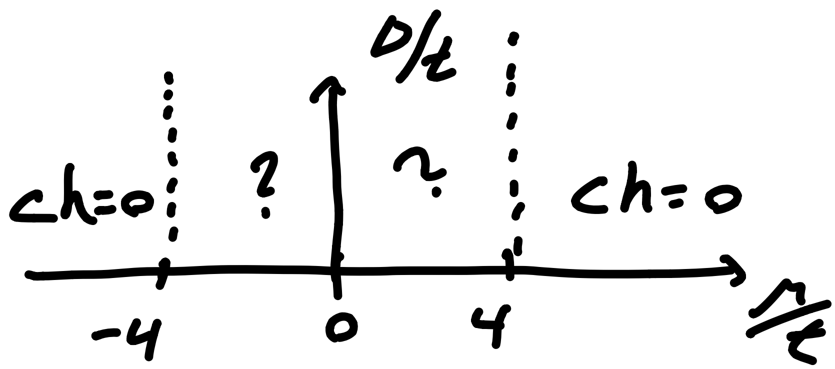
$$\Rightarrow \begin{cases} \vec{k} = (0,0) \Rightarrow \frac{t}{\mu} = -\frac{1}{4} \\ \vec{k} = (0,\pi) \Rightarrow \end{cases}$$

$\mu = 0$

$\vec{k} = (\pi, 0) \Longrightarrow$

$\vec{k} = (\pi, \pi) \Longrightarrow \frac{t}{\mu} = \frac{1}{4}$

c)



i) Can choose Δ, μ, t in each region and calculate:

$$ch = \frac{1}{2\pi} \int d^2k \, F_{x,y}(k)$$

$$F_{x,y}(k) = \frac{1}{2} \varepsilon_{\mu\nu\rho} \hat{d}_\mu \partial_x \hat{d}_\nu \partial_y \hat{d}_\rho$$

$$\hat{d}_\mu = \frac{\vec{d}_\mu}{|d|}$$

$\circledast$ ii) expand around $\mu = 4t$

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at end $\vec{k} = (\pi, \pi) + \vec{p}$

$$\vec{d} = \begin{pmatrix} -2\Delta p_y \\ -2\Delta p_x \\ 4t - \mu \end{pmatrix} \Rightarrow |d| = 2\sqrt{\Delta^2 p_x^2 + \Delta^2 p_y^2 + \left(t - \frac{\mu}{4}\right)^2}$$

$$4t-\mu = -\xi$$

$$\partial_\mu |d| = \frac{4\Delta^2 p_\mu}{|d|}$$

$$\partial_\mu \hat{d}_\nu = \partial_\mu \frac{d_\nu}{|d|} = \frac{\partial_\mu d_\nu}{|d|} - \frac{d_\nu}{|d|^2} \partial_\mu |d|$$

$$= \frac{\partial_\mu d_\nu}{|d|} - \frac{4 d_\nu \Delta^2 p_\mu}{|d|^3}$$

$$\partial_x \hat{d}_x = \frac{8\Delta^3 p_x p_y}{|d|^3}$$

$$\partial_x \hat{d}_y = \frac{-2\Delta |d|^2}{|d|^3} + \frac{8\Delta^3 p_x^2}{|d|^3}$$

$$\partial_x \hat{d}_z = -\frac{4\xi p_x \Delta^2}{|d|^3}$$

$|d|^3$

$$\partial_y \hat{d}_x = -\frac{2\Delta|d|^2}{|d|^3} + \frac{8\Delta^3 p_y^2}{|d|^3}$$

$$\partial_y \hat{d}_y = \frac{8\Delta^3 p_x p_y}{|d|^3}$$

$$\partial_y \hat{d}_z = -\frac{4\varepsilon\Delta^2 p_y}{|d|^3}$$

$$F_{x,y}(p) = \frac{4\varepsilon\Delta^2}{[4\Delta^2 p^2 + \varepsilon^2]^{3/2}}$$

$$ch = \frac{1}{2\pi}\int d^2p \; F_{xy}(p) =$$

$$= \frac{1}{2\pi}\int p\,dp\,d\theta \; F_{xy}(p) = \int_0^\infty p\,dp \; F_{xy}(p)$$

$$= \frac{1}{2}\,\frac{\varepsilon}{|\Delta|}\int_0^\infty dp \frac{p}{\left[p^2 + \left(\frac{\varepsilon}{2\Delta}\right)^2\right]^{3/2}} =$$

$$= \frac{1}{2}\,\frac{\varepsilon}{|\Delta|}\left|\frac{\Delta}{\varepsilon}\right| = \frac{\text{sign}(\varepsilon)}{2}$$

$$\Rightarrow \Delta ch_{\mu=4t} = ch(\varepsilon>0) - ch(\varepsilon<0)$$

$$= 1$$

for $\mu = 0$, $\vec{K}_{\pm} = (0,\pi) + \vec{p}$

$(\pi, 0) + \vec{p}$

$$\vec{d}_{\pm} = \begin{pmatrix} \pm 2\Delta p_y \\ \mp 2\Delta p_x \\ -\mu \end{pmatrix}$$

$$F^{\pm}_{x,y}(p) = -\frac{4\mu\Delta^2}{\left[4p^2\Delta^2 + \mu^2\right]^{3/2}}$$

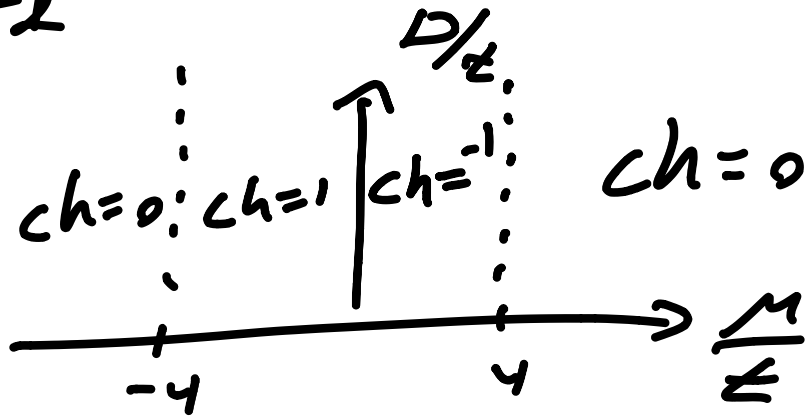
$$ch = -\frac{1}{2}\frac{\mu}{|\Delta|}\int_0^\infty dp\, \frac{p}{p^2 + \left(\frac{\mu}{2\Delta}\right)^2}$$

$$= -\frac{1}{2}\frac{\mu}{|\Delta|}\left|\frac{\Delta}{\mu}\right| = -\frac{1}{2}\,\mathrm{sgn}(\mu)$$

$$\Delta ch = [ch(\mu>0) - ch(\mu<0)] \ldots$$

$\Delta ch = [ch(\mu > 0) - ch(\mu < 0)]^2$

$= -2$



d) $\frac{\mu}{t} = 4 \implies \vec{k} = (\pi, \pi) + \vec{p}$

$$h = \vec{d} \cdot \vec{\sigma} = -2\Delta p_y \sigma_x - 2\Delta p_x \sigma_y - \epsilon \sigma_z$$

$$\implies \alpha = -2\Delta = \beta$$

$$m = -\epsilon = 4t - \mu = -\delta\mu$$

e) $$h(x,y) = i 2\Delta \sigma_x \partial_y + 2\Delta \sigma_y i \partial_x - \epsilon(x) \sigma_z$$

$$h(x,y) \psi(x,y) = 0$$

$$\psi(x,y) = e^{ik_y y}\chi(x)$$

$$\left(-2\Delta k_y \sigma_x + 2\Delta i\sigma_y \partial_x - \epsilon(x)\sigma_z\right]\chi = 0$$

$$2\Delta\, \partial_x \chi = A\chi$$

$$A = -\epsilon(x)\, i\sigma_y \sigma_z - 2\Delta k_y\, i\sigma_y \sigma_x$$

$$= \epsilon(x)\sigma_x - 2\Delta k_y \sigma_z$$

$$= \begin{pmatrix} -2\Delta k_y & \epsilon(x) \\ \epsilon(x) & 2\Delta k_y \end{pmatrix}$$

$$\partial_x U\chi = \frac{1}{2\Delta}\underbrace{UAU^\dagger}_{\Lambda}U\chi$$

simpler: set $k_y = 0$

$$\left(2\Delta i\sigma_y \partial_x - \epsilon(x)\sigma_z\right)\chi = 0$$

$$\Rightarrow \partial_x \chi = \frac{\xi(x)}{2\Delta} \sigma_x \chi$$

$$\Rightarrow \chi_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} / \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\Rightarrow \partial_x \chi_\pm = \pm \frac{\xi(x)}{2\Delta} \chi_\pm$$

$$\chi_\pm = \chi_\pm(x=0)\, e^{\pm \frac{1}{2\Delta}\int_0^x dx\, \xi(x)}$$

$$\xi = m_1 = \begin{cases} m_0 & x>0 \\ -m_0 & x<0 \end{cases}$$

$$\chi_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-\frac{1}{2\Delta}\int_0^x dx\, \xi(x)}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-\frac{m_0}{2\Delta}|x|}$$

$$\psi = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-\frac{m_0}{2\Delta}|x|}\, e^{ik_y y}$$

$$h\psi = E\psi = 2\Delta k_y \psi$$

$E = 2\Delta k_y$

$\xi = \frac{2\Delta}{m_e} \longrightarrow \infty$

⊛ you need only follow

Dirac cones:

$\mu = 4t \Rightarrow -2\Delta(k_y\sigma^x + k_x\sigma^y) + \xi\sigma^z$

rotate: $\sigma_x \to \sigma_y \longrightarrow \sigma_y$

$\sigma_y \to -\sigma_x \longrightarrow \sigma_x$

$\sigma_z \to \sigma_z \to -\sigma_z$

$\Longrightarrow \quad \vec{k}\cdot\vec{\sigma} + \frac{\xi}{2\Delta}\sigma_z = \vec{k}\cdot\vec{\sigma} + m\sigma_z$

$\Delta n = n(m>0) - n(m<0) = 1$

$\Rightarrow n(m<0) = n(\xi<0) =$

$$\underline{= h(m > 0) - 1 = -1}$$

$\mu = 0 \implies \dots$

$\mu = -4t \implies \dots$