Generalized Entanglement as a Framework for Exploring Complex Quantum Dynamics

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What “complexity” and why?

What is complex quantum dynamics? Elusive property...

Complex system: The description follows in principle from a small set of known, basic rules, but the final outcome...is not simple:

- Large size → Inefficient (non scalable) parametrization of system properties...
  e.g., $N$ qubit quantum register, $D = 2^N$
  $|\Psi\rangle = \sum c_{i_1,\ldots,i_N} |i_1,\ldots,i_N\rangle$

- Interaction → Quantum correlations, typically highly non-local...
  e.g., Closed (unitary) dynamics: Many-body quantum systems
  e.g., Open (non-unitary) dynamics: Quantum irreversibility

- Instability → Absence of dynamical regularities...
  e.g., Quantum chaos, non-integrability, disorder

Motivation: Developing a better quantitative understanding of complex quantum behavior is a critical challenge for...

→ Quantum information theory: Benchmarking of QI processors; Decoherence control...
→ Condensed-matter theory: Strongly correlated systems, low-temperature physics, QPTs...
→ Fundamental quantum theory: Correspondence principle; Emergence of classicality...

Can ideas and tools from QI science be used to obtain insight into the properties of complex quantum systems?

[Bennett, 1990; Aharonov, 1999; Nielsen, 1998; Preskill, 1999...]
**Entangled states**: Joint states of a composite quantum system that cannot be expressed as a mixture of products of states of its constituent subsystems.

Entanglement is intimately tied to inherent “complexity” of QI processing:

- Can lead to quantum correlations between subsystems that admit no local classical interpretation (can violate Bell's inequalities...)
- Provides the defining resource for quantum communication (quantum teleportation, superdense coding, communication complexity...)
- Provides a necessary resource for (pure-state) quantum computational speed-up...

\[
\text{Amount of entanglement} \quad \text{upper bounded by} \quad \text{Efficient (poly(n) resources)} \quad \text{classical simulatability}
\]

Pay off for proper accounting of entanglement in complex system already impressive:

- Improved renormalization-group methods for quantum lattice systems in \(D=1,2\) spatial dim
  → "Projected Entangled-Pair States"- enhanced DMRG
  → Entanglement renormalization group

Yet...

Is the current theory of entanglement appropriate/applicable to the full variety of Physics settings?

[Josza & Linden, 2002; Vidal, 2003]

[Porras, Verstraete & Cirac, 2005; Vidal, 2005]
Entanglement and subsystems

Given a quantum system $Q$ with state space $\mathcal{H}$:

- $\mathcal{H}$ can \textit{a priori} support inequivalent tensor product structures. Entanglement of states in $\mathcal{H}$ is unambiguously defined once a \textbf{preferred subsystem decomposition} is chosen:

$$\mathcal{H} \simeq \bigotimes_i \mathcal{H}_i$$

Relative to the selected multipartite structure:

**Entangled states:** A pure $|\psi\rangle \in \mathcal{H}$ is entangled iff $|\psi\rangle$ cannot be prepared by LOCC out of a product state.

A pure $|\psi\rangle \in \mathcal{H}$ is entangled iff $|\psi\rangle$ induces \textbf{mixed subsystem states}. (Def 3)

$$|\psi\rangle = \frac{|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B}{\sqrt{2}}$$

$$\rho_A = Tr_B |\psi\rangle \langle \psi|$$

$$\rho_B = Tr_A |\psi\rangle \langle \psi|$$

Mixed

- The choice of preferred subsystems is usually unproblematic in standard QIS settings.

Spatially separated
local parties

\hspace{2cm}

\hspace{2cm}

Distinguishable/Locally accessible
quantum subsystems

However, local distinguishability assumption is too narrow in other physical settings...
Entanglement in systems subject to superselection rules:

\[ H \approx \bigoplus_N H_N \quad N=0,\ldots,\infty \]

No longer true that separable \( \Rightarrow \) LOCC-preparable... Subsystems do not capture locality...

Entanglement in systems of indistinguishable particles:

Accessible state space is symmetric/antisymmetric subspace of the full tensor product state space

\[ \langle \vec{r}_1, \vec{r}_2, \ldots | \Psi \rangle \sim \begin{vmatrix} e^{i \vec{k}_1 \cdot \vec{r}_1} & e^{i \vec{k}_2 \cdot \vec{r}_1} & \cdots \\ e^{i \vec{k}_1 \cdot \vec{r}_2} & e^{i \vec{k}_2 \cdot \vec{r}_2} & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix}, \quad \langle \vec{r}_1, \vec{r}_2, \ldots | \Psi \rangle = -\langle \vec{r}_2, \vec{r}_1, \ldots | \Psi \rangle \]

No longer true that quantum correlations \( \Rightarrow \) usable resource in the QIP sense...

- **Particles or modes?** Most Slater determinants are mode-entangled, yet the fermions do not interact...
- **Which set of modes?** Inequivalent factorizations may be possible or relevant...
- **What if the algebraic language is changed?** Jordan-Wigner mappings can change fermions into spins...

The choice of preferred subsystems may become problematic in the presence of nontrivial fundamental or operational constraints.
Desiderata for a “generalized” theory of entanglement:

- Consistent with existing theory/results in well-characterized, limiting conditions
- Applicable to arbitrary operator language (fermions, bosons, spins, anyons...)
- Capable to incorporate physical constraints (e.g., limited means to access/control system)

Outline:

I. The notion of Generalized Entanglement (GE)
   A subsystem-independent generalization of entanglement

II. Applications to complex quantum dynamics:

IIa. GE and quantum phase transitions
   Nature and measure of entanglement in quantum phase transitions

IIb. GE and quantum chaos
   Multipartite entanglement generation and fidelity decay in disordered qubit systems
   Generalized entanglement as a natural framework for exploring quantum chaos
Entanglement is an “observer-dependent” concept:

**Keyword:** Define GE relative to a distinguished set of observables $\Omega$ of $S$, rather than a decomposition into subsystems.

- **Steps toward GE:**
  1. Recall that pure entangled states are those for which at least a subsystem state is mixed.
  2. Consider states as positive *linear functionals* on operators:
     \[
     \mathcal{H}\text{-state} \left| \psi \right\rangle : \lambda : \text{End}(\mathcal{H}) \rightarrow \mathbb{C} , \quad \lambda(X) = \text{Tr}(\left| \psi \right\rangle \left\langle \psi \right| X) = \left\langle \psi \right| X \left| \psi \right\rangle
     \]
     A reduced state relative to $\Omega$ is defined only by expectation values of observables in $\Omega$:
     \[
     \Omega\text{-state:} \quad \omega : \Omega \rightarrow \mathbb{C} , \quad \omega = \lambda \mid \Omega
     \]
  3. Observe that the set of $\Omega$-reduced states is convex:
     \[
     x, y \in C \Rightarrow px + (1 - p)y \in C , \quad p \in [0,1]
     \]
     An $\Omega$-reduced state is pure iff it is extremal i.e., it cannot be written as a convex combination of other reduced states.

- **Degree of entanglement directly determined by expectations of observables:**

  A pure state is generalized entangled relative to $\Omega$ if its reduced state is non-extremal (mixed), generalized unentangled otherwise.
**Keyword:** Assume that $\Omega$ is a (semisimple) Lie algebra $\mathfrak{h}$, irreducibly represented in $\mathcal{H}$.

- Natural GE measure: Let \{x\textsubscript{i}\} be a Hermitian, orthogonal basis for $\mathfrak{h}$. Define

$$P_{\mathfrak{h}}(|\psi\rangle) = K \sum_{i} |\langle \psi | x_{i} \rangle |^{2}$$

$K$ is a global normalization factor chosen such that $P_{\mathfrak{h}}^{\text{max}} = 1$ for all G-unentangled $|\psi\rangle$.

- Geometrical meaning:

$$P_{\mathfrak{h}}(|\psi\rangle) = \text{Tr} \left( (\Pi_{\mathfrak{h}}|\psi\rangle \langle \psi|)^{2} \right) = \text{Square length of projection of } |\psi\rangle \langle \psi| \text{ on } \mathfrak{h}.$$ 

- Invariance under group transformations:

$$P_{\mathfrak{h}}(|\psi\rangle) = P_{\mathfrak{h}}(D|\psi\rangle) \quad D = \exp(i \sum_{i} t_{i} x_{i}) \in G, \; t_{i} \in \mathbb{R}$$

A pure state is G-unentangled relative to $\mathfrak{h}$ iff it is a *generalized coherent state* (GCS) relative to the Lie group generated by $\mathfrak{h}$.

- Cartan-Weyl basis: $\mathfrak{h} = \{ \text{CSA} \oplus \mathfrak{h}_{+} \oplus \mathfrak{h}_{-} \}$

$$|\text{GCS}\rangle = \exp \left( \sum_{k} \alpha_{k} A_{k} + \alpha_{k}^{*} A^{*}_{-k} \right) |\text{REF}\rangle$$

GCSs minimize invariant uncertainty

$$\left( \Delta I \right)^{2} = \sum_{i} \left[ \langle x_{i}^{2} \rangle - \langle x_{i} \rangle^{2} \right] = \langle C_{2} \rangle - P_{\mathfrak{h}}$$

"Most classical"/least complex states...
Example I: Multipartite qubit entanglement

**System:** $N$ spin-1/2 particles ($N$ qubits).

→ Local qubit observables are distinguished: $\mathcal{H} = \bigoplus_i \mathfrak{su}(2)_i = \text{span}\{\sigma_{\alpha}^i | \alpha = x, y, z \}$

Standard multipartite entanglement $\equiv$ GE relative to *all* local observables

$$P_{\mathcal{H}}(\left|\psi\right\rangle) = \frac{1}{N} \sum_{i,\alpha} \langle \psi | \sigma^i_{\alpha} | \psi \rangle^2$$

**Theorem:** For every pure state $|\psi\rangle \in (\mathbb{C}^2)^\otimes N$ the following identities hold:

1. The local purity is proportional to the *average subsystem purity*,

   $$P_{\mathcal{H}}(\left|\psi\right\rangle) = \frac{2}{N} \sum_i \left( \text{Tr} \rho_i^2 - \frac{1}{2} \right) = \frac{2}{N} \sum_i \text{Tr} \rho_i^2 - 1$$

2. The local purity is proportional to the amount of *global entanglement* $Q$,

   $$P_{\mathcal{H}}(\left|\psi\right\rangle) = 1 - Q(\left|\psi\right\rangle)$$

**Remarks:**

→ *Every* pure state is *unentangled* relative to *full* algebra $\mathcal{G} = \mathfrak{su}(2^N) = \text{span}\{\sigma_{\alpha}^i \otimes \sigma_{\beta}^2 \otimes ... \otimes \sigma_{\xi}^N \}$

$$|W_N\rangle = \frac{1}{\sqrt{2}} (|\downarrow \uparrow \uparrow ... \uparrow\rangle + |\uparrow \downarrow \uparrow ... \uparrow\rangle + ... |\uparrow \uparrow \uparrow ... \downarrow\rangle) \quad \Rightarrow \quad P_{\mathcal{H}}(|W_N\rangle) = \left(\frac{N-2^2}{N}\right)<1, \quad P_{\mathcal{G}}(|W_N\rangle) = 1$$

→ No unique measure can fully characterize multipartite GE.
**Example II: Indecomposable quantum systems**

**System:** A single spin-1 particle

→ State space $\mathcal{H} \simeq \mathbb{C}^3$:
  
  - Carries the spin-1 irrep of $su(2) = \text{span}\{ J_x, J_y, J_z \}$
  - $\hat{h} = \{ \text{CSA} \oplus \hat{h}_+ \oplus \hat{h}_- \}$, CSA = span{$J_z$}, $\hat{h}_+ = \text{span}\{J_+\}$, $\hat{h}_- = \text{span}\{J_-\}$
  - $| j=1, m=1 \rangle$ is the lowest-weight state.
    
    $$
    J_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}
    $$

→ Assume that distinguished observables are linear in $J_a$: $\hat{h} = su(2)$
  
  - The reduced states can be identified with vectors of expectations of the generators:
    
    $$
    \lambda_{\text{red}} \leftrightarrow \begin{pmatrix} \langle J_x \rangle \\ \langle J_y \rangle \\ \langle J_z \rangle \end{pmatrix} \in \mathbb{R}^3, \quad \text{with} \quad \langle J_x \rangle^2 + \langle J_y \rangle^2 + \langle J_z \rangle^2 \leq 1
    $$
  
  - Pure states are those on the surface = SU(2) angular momentum spin coherent states:
    
    $$
    (n \cdot J) | \xi \rangle = J | \xi \rangle, \quad | \xi \rangle = \exp(\xi J_+ - \xi^* J_-) | 1, -1 \rangle, \quad \xi \in \mathbb{C}
    $$

  - $| 1, -1 \rangle, | 1, 1 \rangle$ are GCSs, $| 1, 0 \rangle$ is not: $| 1, 0 \rangle$ is generalized entangled relative to $su(2)$.
    
    $$
    | 1, 0 \rangle \approx \frac{1}{2} \left( | 1, 1 \rangle \langle 1, 1 | + | 1, -1 \rangle \langle 1, -1 | \right)
    $$
    
    For all practical purposes...

→ All pure states are unentangled relative to $\hat{h} = su(3)$.
**Example III: Fermionic entanglement**

**System:** $N$ spinless fermion modes

\[ \{ c_i, c_j^\dagger \} = \delta_{ij}, \quad \{ c_i, c_j \} = 0, \quad \{ c_i^\dagger, c_j^\dagger \} = 0 \]

→ ... different sites in a lattice...
→ ... delocalized momentum modes... \[ c_j \rightarrow \sum_m U_{mj} c_m, \quad U \in Mat(N \times N), \text{ unitary} \]

- Lie algebra of *number-preserving* fermionic operators:

\[
\hat{h} = u(N) = \text{span} \left\{ c_i^\dagger c_i - \frac{1}{2}, \frac{c_i^\dagger c_j + c_j^\dagger c_i}{\sqrt{2}}, \frac{c_i^\dagger c_j - c_j^\dagger c_i}{i\sqrt{2}} \right\} \quad 1 \leq i < j \leq N
\]

\[
P_{\hat{h}} (|\psi\rangle) = \frac{2}{N} \sum_{j < j', = 1}^N \left[ \langle c_j^\dagger c_j + c_j^\dagger c_j, c_j^\dagger c_j - c_j^\dagger c_j \rangle^2 - \langle c_j^\dagger c_j, -c_j^\dagger c_j \rangle^2 \right] + \frac{4}{N} \sum_{j = 1}^N \left\langle c_j^\dagger c_j - \frac{1}{2} \right\rangle^2
\]

The GCSs of $u(N)$ are the *fermionic product states* = Slater determinants

\[ |\Psi\rangle = \prod_i c_i^\dagger |0\rangle_F = |F \cdot \zeta\rangle_{U(N)} = \exp (\sum_{ij} \zeta_{ij} c_i c_j^\dagger) \]

→ $P_{u(N)} = 1$ for *any* Slater determinant (with any number of fermions);
→ $P_{u(N)} < 1$ for any other pure fermionic state.

*A good measure of fermionic entanglement - independent of set of modes chosen!*
QPTs: Structural change in the ground state as some Hamiltonian parameter is varied, (ideally) at zero temperature.

Can entanglement theory offer a deeper & unified understanding of QPTs?

→ What kind of entanglement is relevant at criticality?
→ What distinctive properties does it have? (scaling behavior)
→ Can entanglement measures reliably diagnose and characterize QPTs?

[Osborne & Nielsen, 2002; Osterloh et al., 2002; Vidal et al., 2003; Latorre et al., 2003; Roscilde et al., 2003; Deng et al., 2003; Wu et al., 2004, 2005; Keating & Mezzadri, 2004; Anfossi et al., 2005; Wolf; Deng et al., 2005. ... ... ...]

Paradigmatic case: Anisotropic XY model in a transverse field

\[ H = -g \sum_{i=1}^{N} \left[ (1 + \gamma) \sigma_x^i \sigma_x^{i+1} + (1 - \gamma) \sigma_y^i \sigma_y^{i+1} \right] + \sum_{i=1}^{N} \sigma_z^i \]

\( g \in [0, \infty) \): spin-spin interaction strength/magnetic field intensity [tunable]
\( \gamma \in [0, 1] \): anisotropy in XY plane \( \Rightarrow \gamma = 0 \): isotropic XY model; \( \gamma = 1 \): Ising model.
Exact solution and criticality properties

Steps:

→ Jordan-Wigner transformation to spinless fermions:

\[ c_j^\dagger = \left( \prod_{j=1}^{N} (-\sigma_z^j) \right) (\sigma_x^j + i\sigma_y^j) \]

→ Fourier transform to momentum modes:

\[ c_k^\dagger = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{-i j k} c_j^\dagger, \quad k \in K \]

→ Bogoliubov quasiparticle transformation:

\[ \gamma_k^\dagger = u_k c_k^\dagger + i v_k c_k, \quad u_k^2 + v_k^2 = 1 \]

The Hamiltonian can be diagonalized exactly:

\[ H = \sum_{k \in K} \epsilon_k (\gamma_k^\dagger \gamma_k - 1/2) \quad \text{with the ground state} \quad |BCS\rangle = \prod_{k > 0} (u_k + i v_k c_k^\dagger c_k^\dagger) |0\rangle_F \]

A second order QPT between a paramagnetic (disorder) state and a ferromagnetic (order) state occurs for \( \gamma > 0 \) at the critical value \( g_c = 1/2 \) in the thermodynamic limit. Spontaneous magnetization (long-range order) for \( g > g_c \) is due to spontaneous symmetry breaking of \( \mathbb{Z}_2 \).
$u(N)$ purity in the anisotropic transverse XY model

- Relevant Lie algebras acting on the $2^N$-dimensional spin space:

$$u(N) = \left\{ c_i^+ c_j, -\frac{1}{2}, \frac{c_i^+ c_j + c_j^+ c_i}{\sqrt{2}}, \frac{c_i^+ c_j - c_j^+ c_i}{i\sqrt{2}} \right\} \quad 1 \leq i < j \leq N$$

$$so(2N) = u(N) \oplus \left\{ \frac{c_i^+ c_j + c_j c_i}{\sqrt{2}}, \frac{c_i^+ c_j - c_j c_i}{i\sqrt{2}} \right\} = u(N) \oplus r \quad \text{Number-non-conserving}$$

- The BCS state is always a GCS of $so(2N)$: $(|BCS\rangle) = \exp\left\{ i\sum_{k>0} \phi_k (c_k^+ c_k + c_k c_k) \right\}|0\rangle_F$

  G-unentangled state, carrying no information about the critical behavior, $P_{so(2N)} = 1$.

- The BCS state becomes a GCS of $u(N)$ in the special case $g = 0$ ($|G\rangle = |\downarrow \downarrow \ldots \rangle = |0\rangle_F$).

- Relative $u(N)$-purity:

$$P_{u(N)}(|BCS\rangle) = \frac{4}{N} \sum_k \left\langle c_k^+ c_k - \frac{1}{2} \right\rangle^2 = \frac{4}{N} \sum_k \left( v_k - \frac{1}{2} \right)^2$$

- In the thermodynamic limit $P_{u(N)}$ behaves as

$$P_{u(N)}(|BCS\rangle) = \begin{cases} 
\frac{1}{1 - y^2} \left( 1 - \frac{y^2}{\sqrt{1 - 4g^2(1 - y^2)}} \right) & g \leq 1/2 \\
\frac{1}{1 + y} & g > 1/2 
\end{cases}$$
One can define a disorder parameter by 
\[ P'_{u(N)} = P_u(N) - \frac{1}{1+\gamma} \]

\[ P'_{u(N)}(\mid BCS \rangle) = \begin{cases} \frac{\gamma}{1-\gamma^2} \left( 1 - \frac{\gamma}{\sqrt{1 - 4 \ g^2 (1 - \gamma^2)}} \right) & g \leq 1/2 \\ 0 & g > 1/2 \end{cases} \]

\[ P'_{u(N)}(\mid BCS \rangle) = g_c - 2g^2 \] (Ising)

\[ \lim_{|i-j| \to \infty} |\langle \sigma_x^i \sigma_x^j \rangle| \sim \exp \left( -\frac{|i-j|}{\epsilon} \right) \]

→ Taylor expansion of \( P'_{u(N)} \) near \( g_c \) implies
\[ \epsilon = const \ (g_c - g)^{-\nu} \]
\( \nu = 1 \)
Ising universality class
Further developments: Quantum complexity

- GE approach successfully applied to other QPTs e.g. detection of both second- and first-order QPTs in the Lipkin-Meshkov-Glick model.

**Emerging picture...**

Identification of relevant algebra of observables is straightforward for “Generalized Mean-Field Hamiltonians”...

\[
H_{GMF} = \sum_{\alpha} \epsilon_{\alpha} A_{\alpha} \equiv \sum_{k=1}^{r} u_{k} h_{k} + \sum_{j=1}^{l} \left( v_{j} e_{\alpha}^{+} + v_{j}^{*} e_{\alpha}^{-} \right), \quad \epsilon_{\alpha}, u_{k} \in \mathbb{R} ; \quad v_{j} \in \mathbb{C}
\]

\[
h = \h_{CSA} \oplus \h^{+} \oplus \h^{-}, \quad M = r + 2l \leq \text{poly}(\log(\dim(\mathcal{H}))) \quad \text{e.g., } \quad H_{XY} \in \text{so}(2N), \quad M=2N^2-N, \quad \dim(\mathcal{H}) = 2^{N}
\]

\[
H_{GMF} \in \h \quad \quad P_{\hat{h}} = \text{const} - \text{no information on criticality}
\]

→ The non-degenerate ground state of $H_{GMF}$ is always a CGS of $\hat{h}$.

\[
H_{GMF} = H_{0} + \Delta H, \quad H_{0} \in \h_{0} \quad \quad \h_{0} \subset \h
\]

→ $P_{\hat{h}_{0}}$ will detect the QPT if the corresponding GE depends on the parameters in $H_{GMF}$.

→ If the ground state is degenerate, $P_{\hat{h}_{0}}$ must be the same for all ground states.

**Theorem:** GMFHs can be efficiently (exactly) solved (\(n \leq \text{poly}(\log(\dim(\mathcal{H})))\) operations).

Somma et al., quant-ph/0601030.
Dynamical crossover to chaos in interacting many-body systems: Structural change in any typical many-body state as interaction strength $J_c$ exceeds threshold value:

$$J_c \approx \left( \text{Average spacing between directly coupled states} \right) \approx \frac{\delta}{\text{No. qubits}}$$

Can entanglement theory offer a deeper & unified understanding of QChaos/QLocalization?

→ How does entanglement behave across a transition to chaos?
→ Can entanglement properties/evolution detect different regimes?

[Lakshminarayan, 2001; Montangero et al., 2003; Santos et al., 2004; Mejia-Monasterio et al., 2005; Weinstein & Hellberg, 2005.
...
...
]

Broader problem:

What are reliable indicators of quantum chaotic behavior in general?

• Static signature: Eigenvalue/eigenvector statistics (RMT). However: Can be basis dependent...

• Dynamic signature: Fidelity decay. However: Can be basis and perturbation dependent...

GE, in principle, is basis-independent and requires no perturbation...
\[ H = \sum_{i=1}^{N} \left[ \Delta_0 + \delta_i \right] \sigma_z^i + \sum_{i,j}^{N} J_{ij} \sigma_x^i \sigma_x^j \]

\( \delta_i \) uniformly random in \([-\delta/2, \delta/2]\)

\( J_{ij} \): uniformly random in \([-J, J]\)

\( J, \delta \ll \Delta_0 \): approximate \( S_z \)-symmetry

\( J > J_c \sim \delta/N \): Fermi Golden Rule regime – Lorenzian LDOS, exponential decay

\( J > J_e \sim \delta / n^{1/4} \): Ergodic regime – Gaussian LDOS, Gaussian decay

- Evolution of local purity starting from computational eigenstate: \( |\psi(0)\rangle = |c\rangle = |0101\ldots01\rangle \)
  \[ P_1(t) = \frac{1}{N} \sum_{i,\alpha} \langle \psi(t) | \sigma_\alpha^i | \psi(t) \rangle^2 \]

- Evolution of bi-local purity starting from entangled eigenstate: \( |\psi(0)\rangle = |01\ldots01\rangle \otimes [ |01\rangle + |10\rangle ]^{\otimes n_s} \)
  \[ P_2(t) = \frac{2}{3N} \sum_{i,k: \alpha \beta} \langle \psi(t) | \sigma_\alpha^i \sigma_\beta^k | \psi(t) \rangle^2 \]

\[ P_k(t_c) = 0.9 \]

- \( n_B = 1 \)
- \( n_B = 2 \)

W-state
Disordered Ising lattice: Further results

- Asymptotic local purity may be estimated from RMT Ansatz:
  - Represent many body state as a *random superposition* of unperturbed states within central band:
    \[ |\psi(\infty)\rangle = \sum_p^D w_p^{\text{RMT}} |p\rangle, \]
    \[ D = \left( \frac{N!}{([N/2]!)} \right)^2 \]
    \[ P_{PT}(|w_p|^2) = (2\pi D |w_\infty|^2) e^{-D |w_p|^2 / 2} \] Porter-Thomas distribution

- Quantify degree of delocalization by *Inverse Participation Ratio*:
  \[ IPR = \frac{1}{\sum_p |w_p|^4} \]

  \[ \langle P_h \rangle \propto \frac{1}{\langle IPR^\infty \rangle} \propto \frac{1}{D} \]

GE directly proportional to delocalization in this regime...
**Case study II: Chaotic quantum maps**

- **Paradigmatic model:** Quantum Kicked Top

  Floquet operator:

  \[ U_{QKT} = e^{-i \frac{\pi J_z}{2} t} e^{-i k \frac{J^2}{2j} t} \]

  \( J\alpha \): angular momentum operators  
  \( k \) : "kick" strength

  \( \rightarrow k < 2.7 \): Non-chaotic (*regular*) classical dynamics;  
  \( \rightarrow 2.7 < k < 4.2 \): Both chaotic and non-chaotic (*mixed*) phase space;  
  \( \rightarrow k > 4.2 \): Fully chaotic dynamics

- **Natural dynamical algebra:** *Spin-J irrep of SU(2)*

  \[ P_{su(2)}(|\psi(t)\rangle) = \frac{1}{J^2} \sum_\alpha \langle \psi(t)|J_\alpha|\psi(t)\rangle^2 \]

  \( \rightarrow \) Explore entanglement generation without coupling to auxiliary system or decomposing into arbitrary subsystems...

  \[ |\psi(0)\rangle = |\xi\rangle, \quad (n \cdot J)|\xi\rangle = J|\xi\rangle, \]

  \[ n = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \quad \text{SU}(2)-\text{GCS} \]

  \( J = 500 \quad k = 3 \)
Why does it work?...

- GE appears related to notion of *quantum extent* of a state relative to observable $A$:

  $\Delta A (|\psi\rangle) = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$

  Peres, in: *Quantum Chaos*, 2001; p.73.

- Square-root of variance of $A$ in state $|\psi\rangle$

- Initial extent of state with respect to *perturbation* related to *fidelity decay*:

  $F(t) = 1 - (\langle V^2 \rangle - \langle V \rangle^2) t^2$ ...

- $GE_{su(2)} = \sum_{l=x,y,z} \Delta \left( \frac{J_l}{J} \right)^2 - \frac{1}{J}$

- GE directly related to *average square extent* for rescaled observables $J_l/J$. 

$k = 1.1$
Conclusions and outlook

- GE provides a **unifying conceptual framework** for defining entanglement relative to any physically relevant, distinguished **subspace of observables**.

- GE is directly applicable to **arbitrary physical settings** and algebraic languages used to describe the system (bosons, fermions, spins etc).

- In the Lie-algebraic setting, generalized-unentangled states are naturally identified with **generalized coherent states**.

- GE provides novel diagnostic tools for **broken-symmetry QPTs** and **quantum chaos**.

*Long list of open problems...*

- Quantum foundations and mathematical formalism (Bell inequalities?)
- Quantum information theory (GLOCC maps)
- Further developments within nilpotent formalism (Classification, dynamics...)
- Quantum many body systems (Topological QPTs, fermionic entanglement...)

- Complex quantum systems (Disordered, nonintegrable, driven...)
- Open quantum systems (Stability of GCSs, dissipative QPTs...)


Thank you for your attention...