Sparsity-based sub-wavelength imaging and super-resolution in time and frequency

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Sub-wavelength images in the microscope
Optical cut-off for high spatial frequencies

\[
\Psi(x, y, z) = \mathcal{F}^{-1}\left\{ \mathcal{F}\{\Psi(x, y, z = 0)\} e^{i\sqrt{\left(\frac{2\pi}{\lambda}\right)^2 - (k_x^2 + k_y^2)}} \right\}
\]

\[
\sqrt{k_x^2 + k_y^2} < \frac{2\pi}{\lambda} \quad \text{propagating waves}
\]

\[
\sqrt{k_x^2 + k_y^2} > \frac{2\pi}{\lambda} \quad \text{evanescent waves}
\]
Hardware solutions for sub-wavelength imaging

- Scanning near-field optical microscope
- Methods using fluorescent particles
- Structured Illumination
- Negative-index / metamaterials structures: superlens, hyperlens
- Hot-spot methods: nano-hole array, super-oscillations

Require scanning, averaging over multiple experiments

Is it possible to have real-time, single exposure sub-wavelength imaging using a ‘regular’ microscope?
Analytic Continuation

• The 2D Fourier transform of a spatially bounded function is an analytic function.

\[ \tilde{\Psi}(k_x, k_y) = FT \left\{ \Psi(x, y, z = 0) \right\} e^{i k_z z} \]

If \( \tilde{\Psi}(k_x, k_y) \) is known at a small region \( k_x, k_y \) in space with infinite accuracy, then \( \tilde{\Psi}(k_x, k_y) \) is known at every point in space.

• **Problem:** Existing analytic continuation methods are not very robust:
  - sampling theorem based extrapolations yield a highly ill posed matrix.
  - Iterative methods (Gerchberg - Papoulis) are sensitive to noise
Common wisdom

“All methods for extrapolating bandwidth beyond the diffraction limit are known to be extremely sensitive to both

• noise in the measured data and

• the accuracy of the assumed a priori knowledge.”

“It is generally agreed that the Rayleigh diffraction limit represents a practical frontier that cannot be overcome with a conventional imaging system.”

Bandwidth extrapolation problem: infinite number of possible solutions!

How to choose the right one?
Problem: non-invertible filter

\[ \begin{align*}
\text{Measurements} & = \text{Filtered Fourier transform (invertible)} \quad \text{signal} \\
\begin{bmatrix}
y_4 \\
y_3 \\
y_2 \\
y_1 \\
y_0 \\
y_{-1} \\
y_{-2} \\
y_{-3} \\
y_{-4} \\ \vdots
\end{bmatrix} & = \begin{bmatrix}
A_{-4-4} & A_{-4-3} & A_{-4-2} & A_{-4-1} & A_{-40} & A_{-41} & A_{-42} & A_{-43} & A_{-44} \\
A_{-3-4} & A_{-3-3} & A_{-3-2} & A_{-3-1} & A_{-30} & A_{-31} & A_{-32} & A_{-33} & A_{-34} \\
A_{-2-4} & A_{-2-3} & A_{-2-2} & A_{-2-1} & A_{-20} & A_{-21} & A_{-22} & A_{-23} & A_{-24} \\
A_{-1-4} & A_{-1-3} & A_{-1-2} & A_{-1-1} & A_{-10} & A_{-11} & A_{-12} & A_{-13} & A_{-14} \\
A_{0-4} & A_{0-3} & A_{0-2} & A_{0-1} & A_{00} & A_{01} & A_{02} & A_{03} & A_{04} \\
A_{1-4} & A_{1-3} & A_{1-2} & A_{1-1} & A_{10} & A_{11} & A_{12} & A_{13} & A_{14} \\
A_{2-4} & A_{2-3} & A_{2-2} & A_{2-1} & A_{20} & A_{21} & A_{22} & A_{23} & A_{24} \\
A_{3-4} & A_{3-3} & A_{3-2} & A_{3-1} & A_{30} & A_{31} & A_{32} & A_{33} & A_{34} \\
A_{4-4} & A_{4-3} & A_{4-2} & A_{4-1} & A_{40} & A_{41} & A_{42} & A_{43} & A_{44} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix} \begin{bmatrix}
x_4 \\
x_3 \\
x_2 \\
x_1 \\
x_0 \\
x_{-1} \\
x_{-2} \\
x_{-3} \\
x_{-4} \\ \vdots
\end{bmatrix}
\end{align*} \]

(buried in the noise)
(exponentially small evanescent waves)
Under-determined system of equations

- Under-determined system of equations: more variables than equations
  \[ Ax = y \]

- Infinite number of solutions (x)
- Choose the one that “makes the most sense”

We choose the solution with maximum sparsity – the one with the fewest nonzero elements.

the object is sparse in a known basis.
Why sparsity?

• **General**: Many objects are sparse in some (general) basis.

• **Powerful**:
  
  • Robust to noise. Without noise, in a sparse enough case the sparsest solution is unique.
  
  • Sparsity is used successfully for image denoising, deconvolution, compression, enhancement of MR images and more. However – has never been used for sub-λ imaging, or temporal bandwidth extrapolation.

• **Attainable**: Efficient algorithms exist for estimating the sparsest solution.
Sparsity – a general feature of information

Sparsity in real space image

- **biological species:**
  - Real-space sparsity \( \sim 2\%-5\% \)

Sparsity in another basis

- **Electronic chips:**
  - Sparsity in gradient basis \( \sim \) few %
How to do it: for example - Basis Pursuit

Solve the (convex) optimization problem:

\[
\min_x \|x\|_1 \quad \text{subject to} \quad \|Ax - y\|_2 < \varepsilon
\]

- \(x\): unknown image
- \(y\): measured image
- \(A\): Low-pass filter + sparsity basis
- \(\varepsilon\): Noise parameter

\[\|x\|_1 = \sum_i |x_i|\]

- The requirement on the \(l_1\) norm is to promote sparsity.
- Find the \textbf{sparsest} \(x\) that is \textbf{consistent with the measurements}.

Proof of concept

Proof of concept

Original

Recovered


**True sub-\(\lambda\) experiments – 1D @ \(\lambda = 532\) nm**

- **Width:** 150 nm
- **Length:** 20 µm
- **Spacing:**
  - 150 nm (left/right pair)
  - 300 nm (center) ~ diffraction limit

**Fabrication:**
Kley – group
University of Jena
kley@iap.uni-jena.de
Best possible microscope image (NA ≈ 1)
Microscope image far-field
Experimental result (with hand-made microscope)

150 nm
Comparison original - reconstruction

real space

spatial spectrum

amplitude [a.u.]

width [nm]

spectral amplitude [a.u.]

spatial frequency [1/\lambda]
True sub-$\lambda$ experiments – 2D @ $\lambda = 532$ nm
Best possible microscope image (NA ≈ 1)
Microscope image far-field
Loss of power in the far-field

more than 90% of the intensity is lost
Experimental results

reconstructed image

SEM image

100 nm
Abbe limit
Can we do sub-wavelength reconstruction based on intensity measurements only? Without measuring phase at all?

Yes, indeed. The knowledge of sparsity is powerful.

First: Fourier phase recovery using iterative algorithm* – given the blurred image intensity and Fourier intensity.

Second: sparsity-based reconstruction using recovered phase.

or, better, combine the two!

Experimental: sparsity-based recovery of ‘random’ distribution of circles

Circles are 100 nm diameter

Wavelength ~ 532 nm

Diffraction-limited (low frequency) intensity measurements

Model

Fourier transform

* Assuming non-negativity
Experimental: incorrect reconstruction with wrong number of circles

30 circles left

22 circles left

11 circles left

12 circles left
Sparsity-based super-resolution in pulse-shape measurements – experimental

\[ V_{osc}(t) = \int IRF(t - t')I_{Laser}(t')dt' \]

Impulse response functions

\[ V_{osc}(\omega) = T(\omega)I_{Laser}(\omega) \]

Transfer functions
Sparsity-based super-resolution in pulse-shape measurements – experimental

Measured signals

Spectra of measured signals

Reconstruction
Because the interferogram cannot be collected from $x = -\infty$ to $+\infty$, it is always truncated, hence some error arises in the resulting spectrum: the spectral line is broadened + side-lobes are added.

Resolution of a F-T spectrometer:

$$\Delta \lambda = 1 / (\text{path difference} = 4x)$$
Sparsity-based super-resolution in FTIR spectrum measurement – experimental example
Conclusions

- method for recovering sub-\(\lambda\) information from the optical far-field of images
- requires *no* additional hardware
- works in *real time and with ultrashort pulses*
- applicable to *all* microscopes (optical and non-optical)
- reconstruction also with incoherent / partially coherent light
- Ideas are universal: can be used to recover
  - shapes of ultrashort pulses in time
  - spectral features
  - *quantum info!*
Many thanks for your attention!
A little about uniqueness

An object comprising on \( n \) ‘features’ is uniquely determined by \( 2n(n+1) \) measurements on a polar grid in k-space, without noise.

## Comparison of approaches

<table>
<thead>
<tr>
<th>Original CS approach</th>
<th>Our CS-related approach</th>
</tr>
</thead>
<tbody>
<tr>
<td>• measurement in uncorrelated basis (commonly Fourier basis)</td>
<td>• measurement in far-field (= Fourier basis) OR blurred near field or in between</td>
</tr>
<tr>
<td>• sampling (randomly) over the entire measurement basis with low resolution</td>
<td>• sampling in a small part of the measurement basis (k_x &lt; k) with high resolution</td>
</tr>
<tr>
<td>• reduction of required samples to retrieve the function</td>
<td>• obtain maximal info on the frequency region where we do NOT measure</td>
</tr>
</tbody>
</table>

We do NOT do CS. We do NOT use CS “rules”. Why does it work for us?
Unique sparse solution

\[ y = Wd_1 = Wd_2 \quad \Rightarrow \quad W(d_1 - d_2) = Wz = 0 \]

triangle inequality:
\[ \|d_1 - d_2\|_0 \leq \|d_1\|_0 + \|d_2\|_0 \quad \Rightarrow \quad \|z\|_0 \leq S_1 + S_2 \]

if every \( S_1 + S_2 \) columns of \( W \) are linearly independant

\[ Wz = 0 \quad \Rightarrow \quad z = 0 \quad \forall z \quad \Rightarrow \quad d_1 = d_2 \]

if \( \|d\|_0 \leq \frac{1}{2} \left( 1 + \frac{1}{\mu(W)} \right) \), then there is a unique sparse solution

\[ \mu \text{ matrix coherence} \]
Reconstruction of the phase (Fienup-Algorithm)

Experimental: holes on a grid

 Recovered image

 Model phase

 Recovered phase

 139 nm

Nonlinear Optics Laboratory
Consider a function $f(t) \in \mathbb{R}^N$ that can be written as a superposition of spikes:

$$f(t) = \sum_{\tau \in T} f(\tau) \delta(t - \tau)$$

If it is comprises of $|T|$ spikes, and $N$ is a prime number, then $f(t)$ can be uniquely defined by any $2|T|$ of its Fourier measurements, defined as:

$$f(\omega) = \sum_{t=0}^{N-1} f(t) e^{-i2\pi\omega t/N} \quad \omega = 0, 1, \ldots, N-1$$

Specifically, the $2|T|$ low pass Fourier coefficients will do.

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Sparsity-based super-resolution in pulse-shape measurements – theoretical example
Sparsity-based super-resolution in pulse-shape measurements – theoretical example

\[ I_{\text{Laser}}(\omega) = \frac{V_{\text{osc}}(\omega)}{T(\omega)} \]
Sparsity-based super-resolution in pulse-shape measurements – theoretical example
Sparsity-based super-resolution in pulse-shape measurements – theoretical example

- 40 ps features are well reconstructed (τ~1 ns)
- Resolution is enhanced by >10 times vs. Wiener de-convolution