Recent observations of Planetary (Rossby) waves' propagation and a new theoretical approach

Nathan Paldor
Department of Atmospheric Sciences
The Hebrew University of Jerusalem

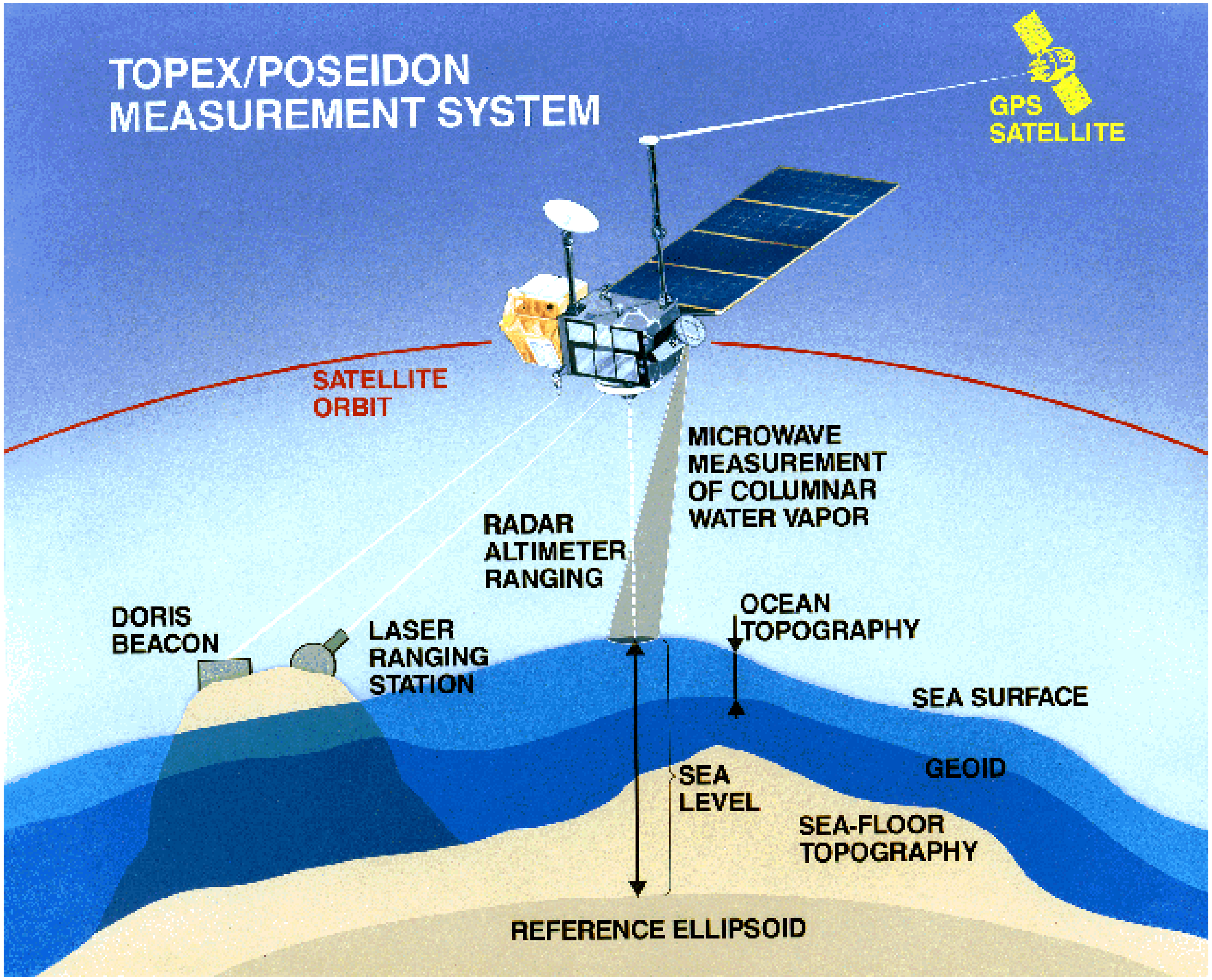
ISEEQ – May/2005

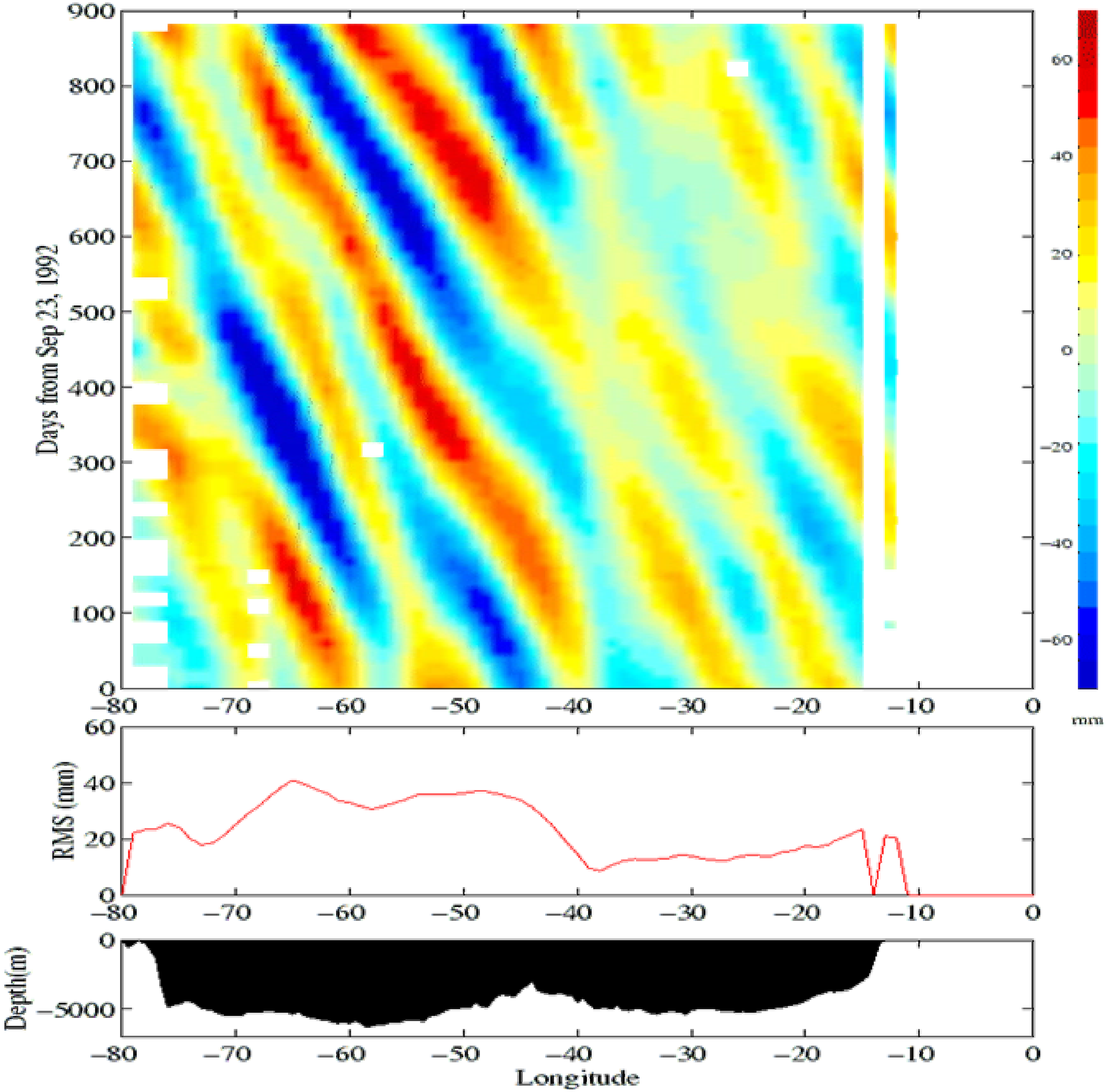
Layout

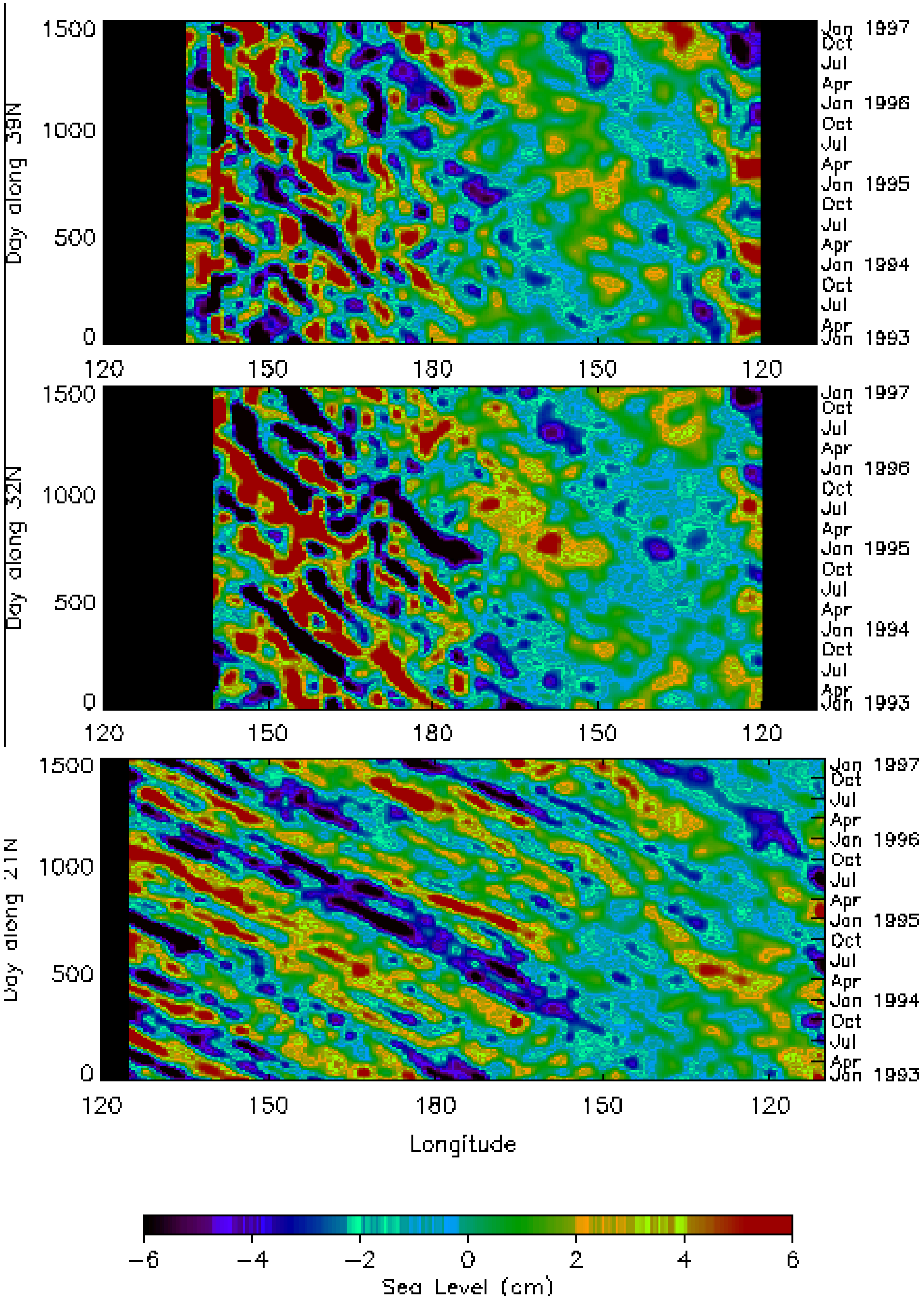
- The observational impetus: Topex/Poseidon observations of crests/troughs of SSH that propagate westward 2-4 faster than expected from wave theory
- Classical wave theory on the β-plane
  - High frequency for constant f (gravity and Inertia-gravity)
  - Low frequency for varying $f=f_0+\beta y$ (Planetary=Rossby)
- A higher order theory of linear waves on a rotating plane based on Schrödinger equation formulation
- Faster phase speed of planetary (Rossby) waves result from the new formulation in Oceanographic regime

S. Rubin

Sea Surface Height propagation observed with the altimeter aboard Topex/Poseidon




Days from Sep 23, 1992
900
800
700
600
500
400
300
200
100
0
−80
−70
−60
−50
−40
−30
−20
−10
0
60
40
20
0
−20
−40
−60
mm
RMS (mm)
60
40
20
0
Depth(m)
0
−5000
Longitude


Day along 39N
Day along 32N
Day along 21N
0
500
1000
1500
120
150
180
150
120
Jan 1993
Apr
Jul
Oct
Jan 1994
Apr
Jul
Oct
Jan 1995
Apr
Jul
Oct
Jan 1996
Apr
Jul
Oct
Jan 1997
Longitude
−6
−4
−2
0
2
4
6
Sea Level (cm)

Interpreting T/P observations: low frequency planetary (Rossby) waves in the thermocline

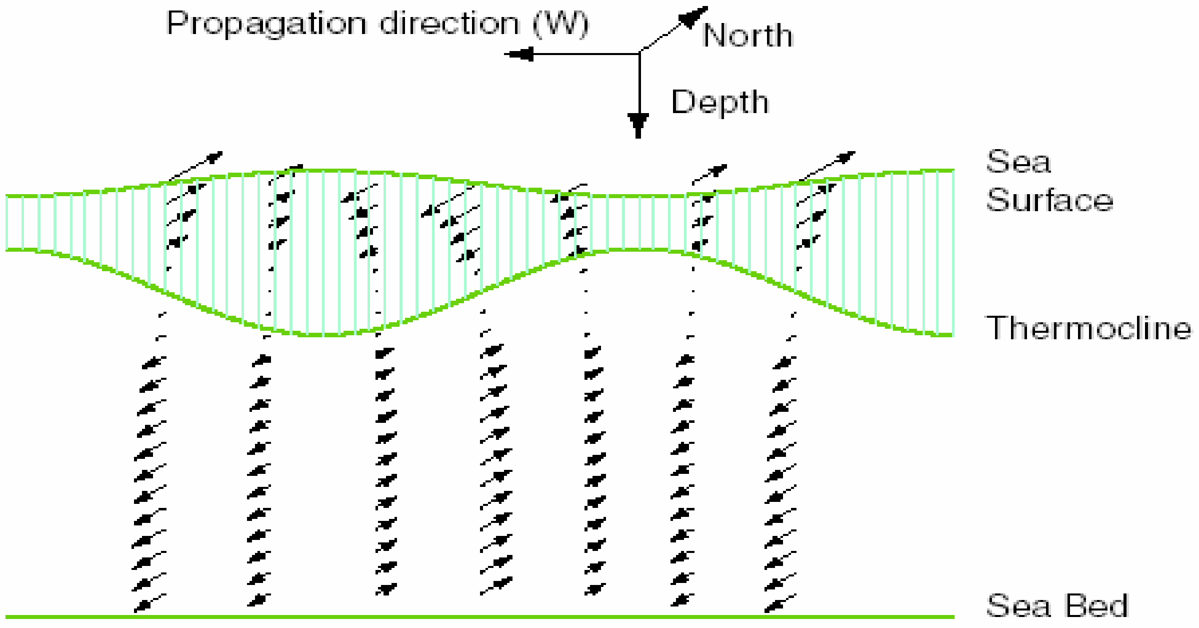


Fig. 1 A schematic of the depth structure of a first-order baroclinic Rossby wave, with a rise in the sea surface of a few centimetres being mirrored at the thermocline by a depression of tens of metres. The meridional velocities change sign about the thermocline.

Other data corroborate T/P findings on SSH

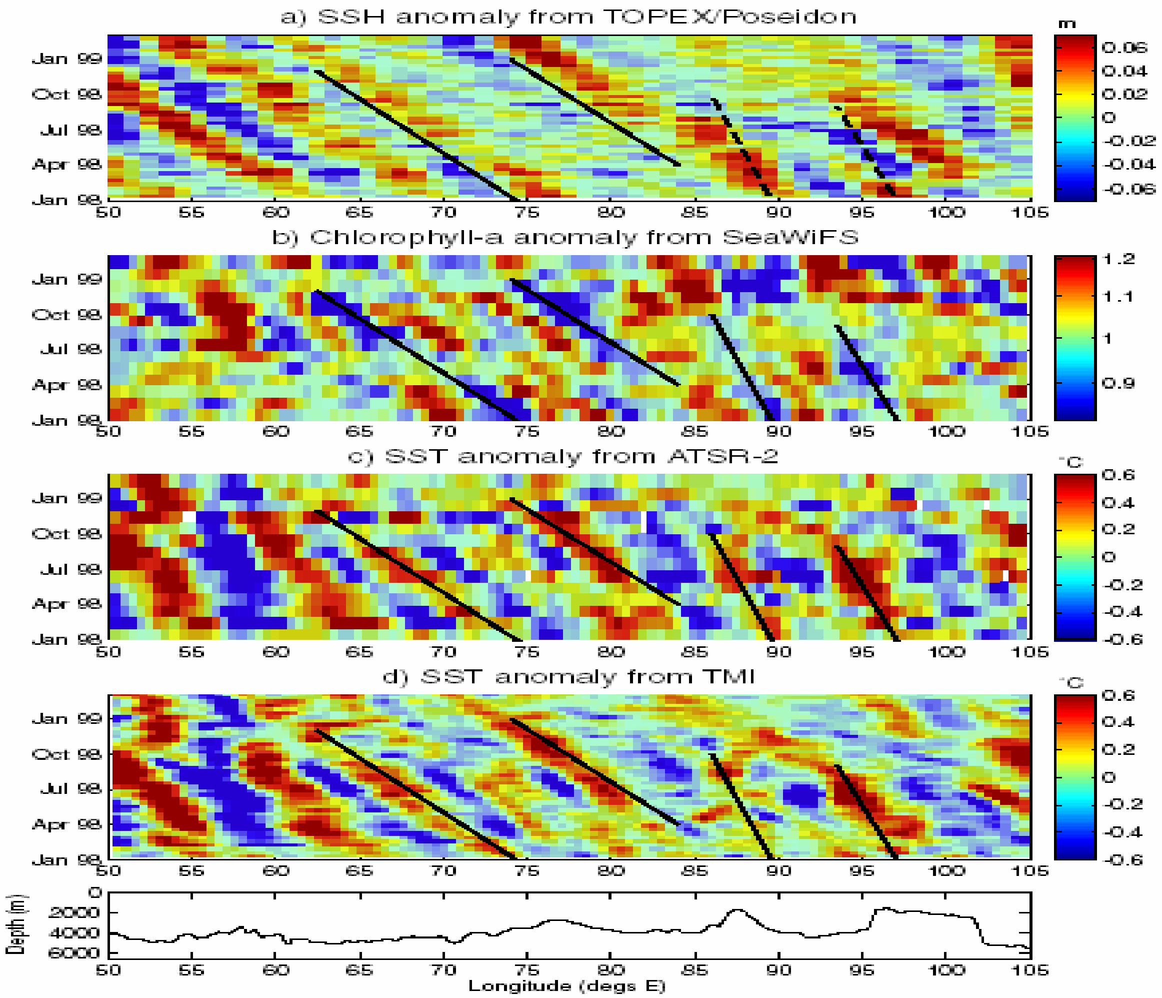
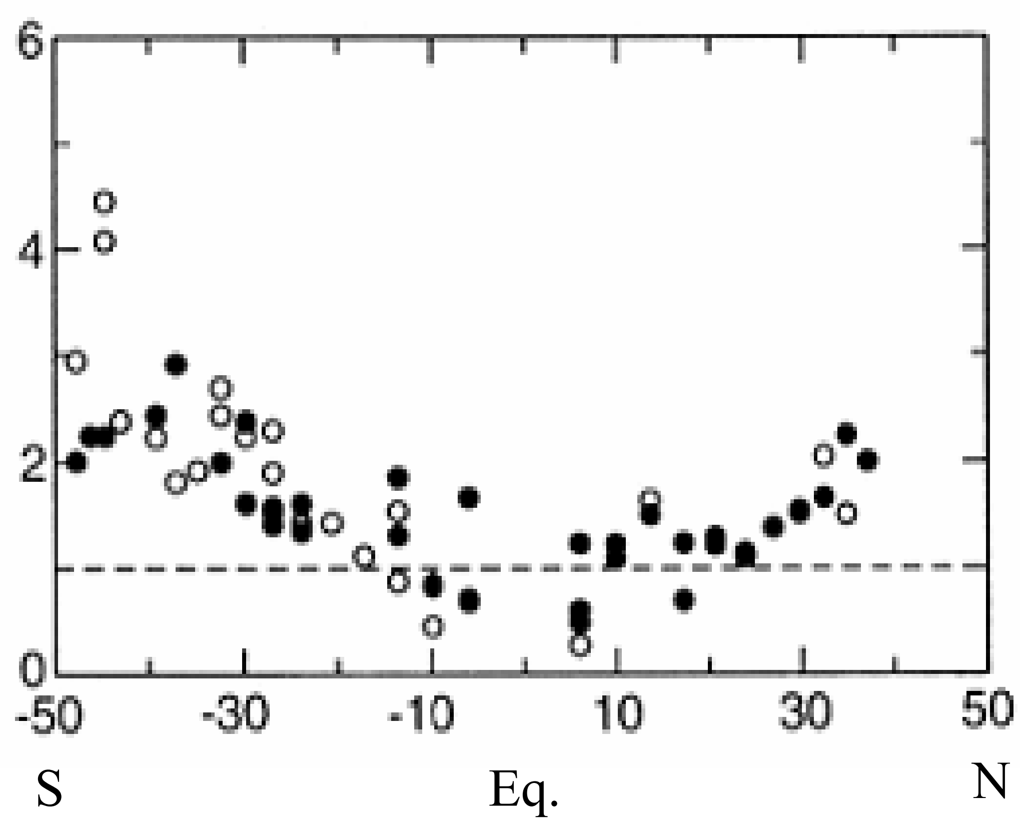


Fig. 3: Hovmöller diagrams of anomalies in SSH, chlorophyll content and SST at 32°S in the Indian Ocean. (The values for chlorophyll content show proportional change.) The added lines (placed along reductions in chlorophyll) highlight common features, with the lines dashed when features are not obviously present in a particular plot. The bottom plot shown the bathymetry along the section.

Topex/Poseidon observations of westward propagation speed of low frequency, long, planetary waves (deSzoeke and Chelton)



Ratio of observed-to-theoretical phase speeds in all oceans

Phase speed of Rossby waves in the N. Atlantic Ocean

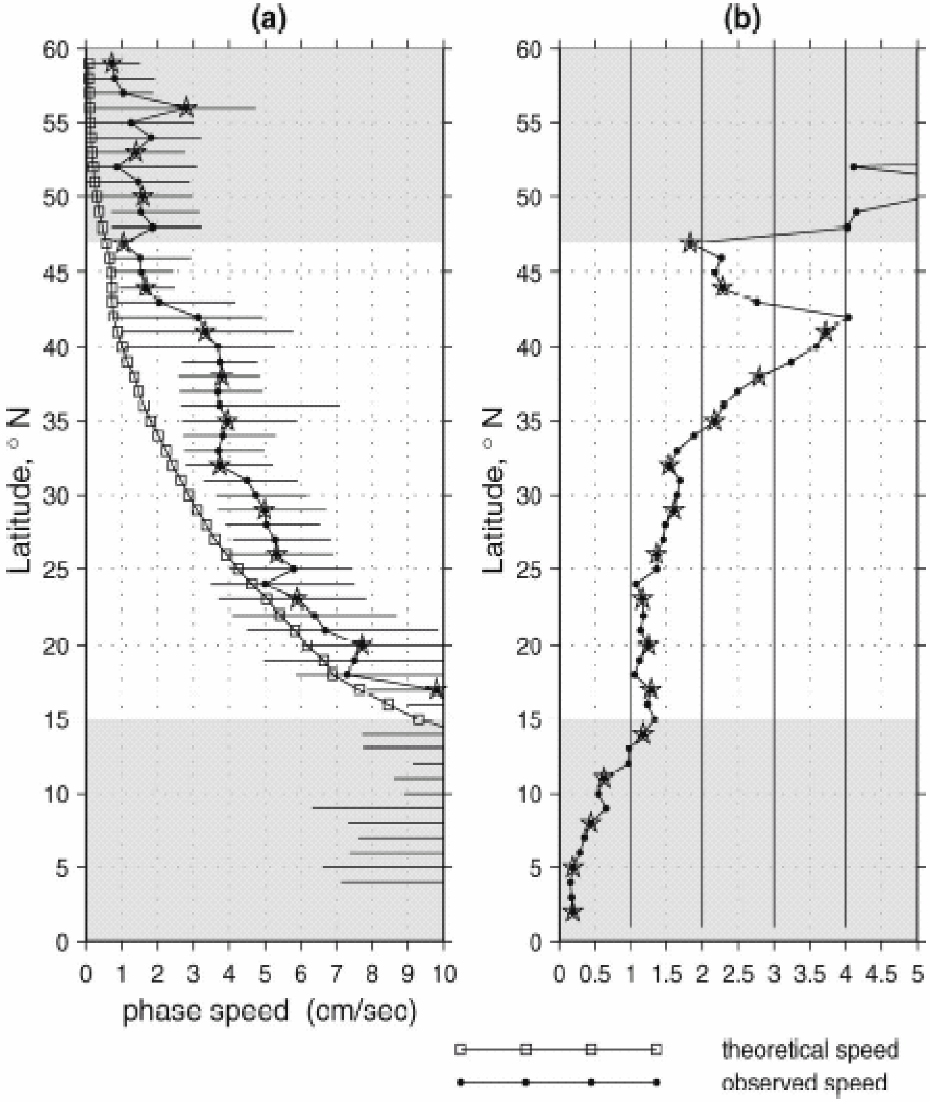


FIG. 7. Comparison of (a) Rossby wave phase speed and (b) ratio of the best-fit Rossby wave phase speed with that expected from theory. See text for more details. Stars denote independent estimates from SSH spectra. Uncertainties estimated from 95% confidence level for a peak in an integrated spectrum (see the appendix). Accurate results are obtained for 15°–47°N. Regions of unreliable estimates are shaded.

(a) Theoretical & observed speeds. (b) Ratio of observed-to-theoretical speeds.

Accurate results are obtained in the 15° – 47°N latitude band. Unreliable estimates are shaded

Osychny & Cornillon (2004)

Topex/Poseidon observations:

- Phase-speed of mid-latitude SSH features **always** exceeds the estimates of Rossby's 1939 theory
- The difference is 100% - 300% (a factor of 2-4!)
- The underestimate of Rossby's theory seems to increase with latitude (but large fluctuations exist)

Q1: Can a higher orders theory be constructed and what is the relevant expansion parameter?

Q2: If so, what is the origin of the difference?

Linearized, planar, Shallow Water Equations with Coriolis force (due to rotation) for **V** and η

$$\frac{\partial \vec{V}}{\partial t} + f\hat{k} \times \vec{V} = -g\nabla \eta$$

$$\frac{\partial \eta}{\partial t} = -H\nabla \cdot \vec{V}$$

where f=2Ωsin(ϕ) and H is the mean height of the fluid

The scalar form in Cartesian coordinates for $f=f_0+\beta y$ is:

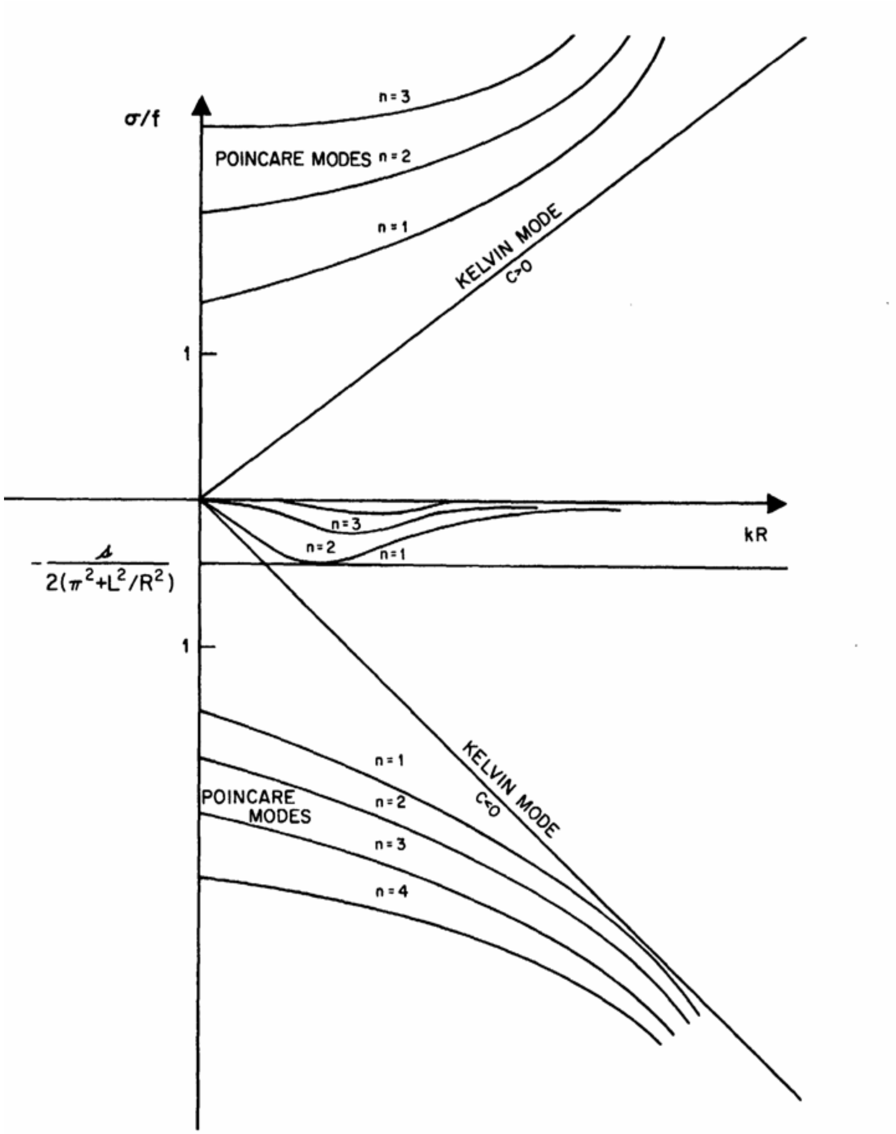
$$\frac{\partial u}{\partial t} - (f_0 + \beta y)v = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + (f_0 + \beta y)u = -g\frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial (Hu)}{\partial x} + \frac{\partial (Hv)}{\partial y} = 0$$

where $f_0=2\Omega\sin(\phi_0)$; $\beta=df/dy=(1/R)df/d\phi=2\Omega\cos(\phi_0)/R$

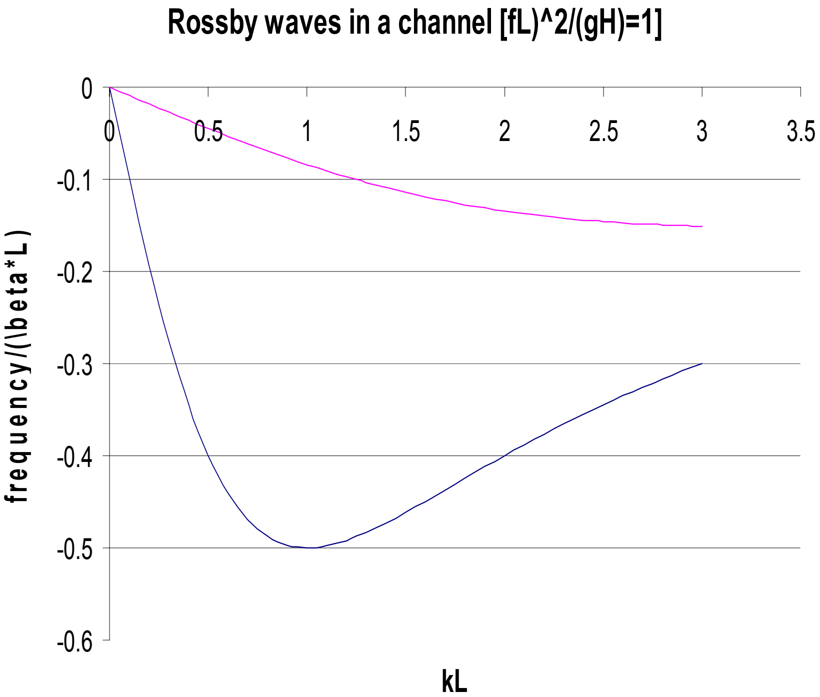
Classical dispersion relations of linear waves in a channel on a rotating (f- or β-) plane (Pedlosky, 1979)



Three types of waves

- Non-dispersive (Kelvin):
  $\sigma = \pm(gH)^{1/2}k$; $v(y) \equiv 0$
  $(gH)^{1/2}$ is the "sound" speed
  (f is in the eigenfunctions)
- Inertia-gravity (Poincare):
  $\sigma = \pm(f^2+(gH)k^2)^{1/2}$
- Planetary (Rossby):
  westward directed phase speed only ($C=\sigma/k<0$; linear with $\beta \equiv df/dy$)

Rossby waves’ dispersion on the β-plane in a zonal channel of width L



$$\sigma=-\beta k/(k^2+(m\pi/L)^2+f^2/gH)$$

1. [mπ/L, k] = [cross-, long-]channel wavenumber (m = ±1, ±2,…)
2. Propagate westward!
3. Derived from vorticity dynamics

   ⇒“nearly” non-divergent flow

   (⇒“rotational” designation)

   However, non-divergence is essential for time-dependence
4. Although $\beta\neq0$, f ($\equiv f_0+\beta y$) is replaced by f_0 everywhere

We wish to construct a new theory that addresses the following shortcomings of the classical theory:

1. Can the full f(y) ($=f_0+\beta y$) variation be included everywhere and what are the resulting phase speeds of the low frequency (Rossby) waves?
2. How does the new estimate of Rossby waves' phase speed compare with T/P observations?
3. High-frequency waves (Gravity and Inertia-gravity) result from the equations in which $f=f_0$. How do they change when $f=f(y)$ is assumed?

Linearized Shallow Water Equations on the β-plane

$$\frac{\partial u}{\partial t} - (f_0 + \beta y)v = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + (f_0 + \beta y)u = -g\frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial (Hu)}{\partial x} + \frac{\partial (Hv)}{\partial y} = 0$$

- Scale length on R, time on 1/(2Ω), velocities on 2ΩR and η on H. Note: $y_{nondim}=y_{dim}/R=\phi-\phi_0$
- Assume a zonally propagating wave solution: $e^{ik(x-Ct)}$ (C is the phase speed; k is the wavenumber (kC=σ)
- Eliminate u (=linear combination of V=i*v*/k and η)
- Get two, 1st order, differential equations for V_y and η_y

$$\frac{dV}{dy}=\left(\frac{\sin(\phi_o)+\cos(\phi_0)y}{C}\right)V+\left(\cancel{\frac{\alpha-C^2}{C}}\right)\eta$$

$$\frac{d\eta}{dy}=\left(\frac{k^2C}{\alpha}-\cancel{\frac{(\sin(\phi_o)+\cos(\phi_0)y)^2}{\alpha C}}\right)V-\left(\frac{\sin(\phi_o)+\cos(\phi_0)y}{C}\right)\eta$$

$\underline{C^2=\alpha}$: (Kelvin) the V-equation decouples and its solution is $V(y)\equiv 0$. $\eta(y)$ is exponential with $-(\sin(\phi_0)+\cos(\phi_0)y)^2/C$ so $\eta(y)$ decreases with +y for $C > 0$ and with -y for $C<0$

$\underline{C^2\neq\alpha}$: The 2, 1st order, equations for V_y and η_y transform into a single, 2nd order, equation for V_{yy} ($=V_{\phi\phi}$)

$$\frac{d^2V}{dy^2} = \left(\frac{\cos(\phi_0)}{C} + \left(k^2 + \frac{(\sin(\phi_0) + \cos(\phi_0)y)^2}{\alpha}\right) - \frac{k^2C^2}{\alpha}\right)V; \qquad V(y=\pm\delta\phi)=0$$

where $\alpha = gH/(2\Omega R)^2$ is a free parameter of the equation

Two complications:

1. y-dependence via the βy term in Coriolis frequency
2. Nonlinearity in C (recall, in Rossby waves $|C| << \alpha^{\frac{1}{2}} < 1$)

Traditional theory: neglect βy ⇒ constant coefficients ⇒

$d^2/dy^2 = -(n+1)^2\pi^2/(2\delta\phi)^2$, n=0, 1,…..

Rossby waves: neglect the C^2 term and get:

$$\boxed{C = -\frac{\cos(\phi_0)}{\left(\frac{(n+1)\pi}{2\delta\phi}\right)^2 + k^2 + \frac{\sin^2(\phi_0)}{\alpha}}}$$

A higher order theory on the β-plane

Set $z=y/\delta\phi \Rightarrow V=0$ BCs are applied at $z=\pm 1$ for all $\delta\phi$

The 2nd order eigenvalue problem in y transforms to:

$$\boxed{\varepsilon^2 \frac{d^2V}{dz^2} + [E-(1+bz)^2]V = 0;} \qquad V(z=\pm 1)=0$$

where:

$$E = -\frac{\alpha\cos(\phi_0)}{C\sin^2(\phi_0)} - \frac{k^2}{\sin^2(\phi_0)}(\alpha - C^2)$$

$$\varepsilon = \frac{\sqrt{\alpha}}{\delta\phi\sin(\phi_0)}$$

$$b = \delta\phi\cot(\phi_0)$$

E: Eigenvalue

$\varepsilon \sim (g'H')^{\frac{1}{2}}$

b: $\max(\Delta f)/f_0$

A **<u>linear</u>** eigenvalue (Schrödinger) equation for $E(\varepsilon, b)$ (no k or C) with potential: $(1+bz)^2$ (“nearly” harmonic oscillator)

Qualitative consequences:

1. For any $\varepsilon=\alpha^{\frac{1}{2}}/(\delta\phi\sin(\phi_0))$ and $b=\delta\phi\cot(\phi_0)$ Sturm-Liouville theorem states that there exist infinitely many eigenvalues E_n ($n=0, 1, 2,\ldots$), all of which are positive and $E_n \sim n^2$, $n\rightarrow\infty$ (regardless of k or C). The 3 roots of the cubic C(E) equation yield the different phase speeds.
2. In principle, standard procedures (e.g. WKB) exist for approximating E_n and $V_n(z)$
3. For each $E_n(\varepsilon, b)$ and $V_n(Z)$ there exist a small-C root of the cubic C(E) equation ($C^2<<\alpha^{\frac{1}{2}}<<C^{-1}$) - Rossby modes:

$$C_n \approx -\cos(\phi_0)[k^2+E_n\sin^2(\phi_0)/\alpha]^{-1}$$

4. For each $E_n(\varepsilon, b)$ and $V_n(Z)$ the cubic C(E) equation has 2 large-C roots ($C^2>>\alpha^{\frac{1}{2}}>>C^{-1}$) - Inertia-gravity modes:

$$C_n^2 \approx \alpha+E_n\sin^2(\phi_0)/k^2$$

5. Kelvin modes are the trivial solution of the eigenvalue equation, $V(y)=0=E$, but with $C^2=\alpha$ and $\eta(y)\neq 0$

$E_n(\varepsilon, b=0)$ (Rossby's original theory)

For b=0 the ODE has constant coefficients:

$$\varepsilon^2 V_{zz} + (E-1)V=0; \qquad V(z=\pm 1)=0$$

Exact solutions for the eigenavlues (only $E_n > 1$ are allowed!) are:

$$E_n = 1 + \frac{[(n+1)\varepsilon\pi]^2}{4}; \qquad n = 0,1,2\ldots$$

The low ($C^2 \ll \alpha$) phase speed, C_n, from this expression for E_n is:

$$\boxed{C_n = \frac{-\cos(\phi_0)}{k^2 + E_n \dfrac{\sin^2(\phi_0)}{\alpha}} = \frac{-\cos(\phi_0)}{k^2 + \dfrac{\sin^2(\phi_0)}{\alpha} + \dfrac{(n+1)^2 \pi^2}{(2\delta\phi)^2}}}$$

The $E_n(\varepsilon, b=1)$ case

Symmetry of quadratic potential $(1+1\cdot z)^2$ about z=-1 yields exact solutions of the Schrödinger Eqn. with eigenvalues E_n:

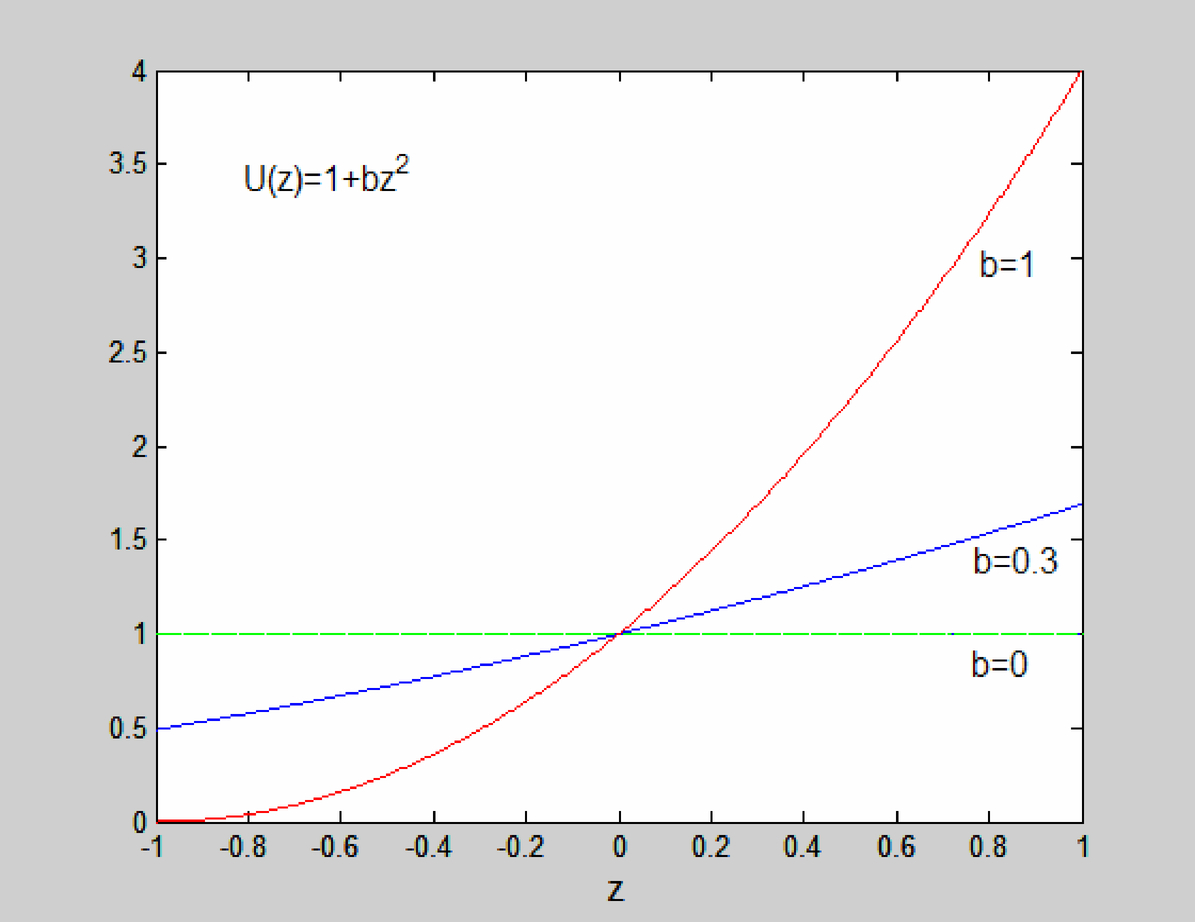
$$E_n = (4n+3)\varepsilon; \qquad n = 0,1,2\ldots$$

and in particular: $E_0(\varepsilon, b=1)=3\varepsilon$

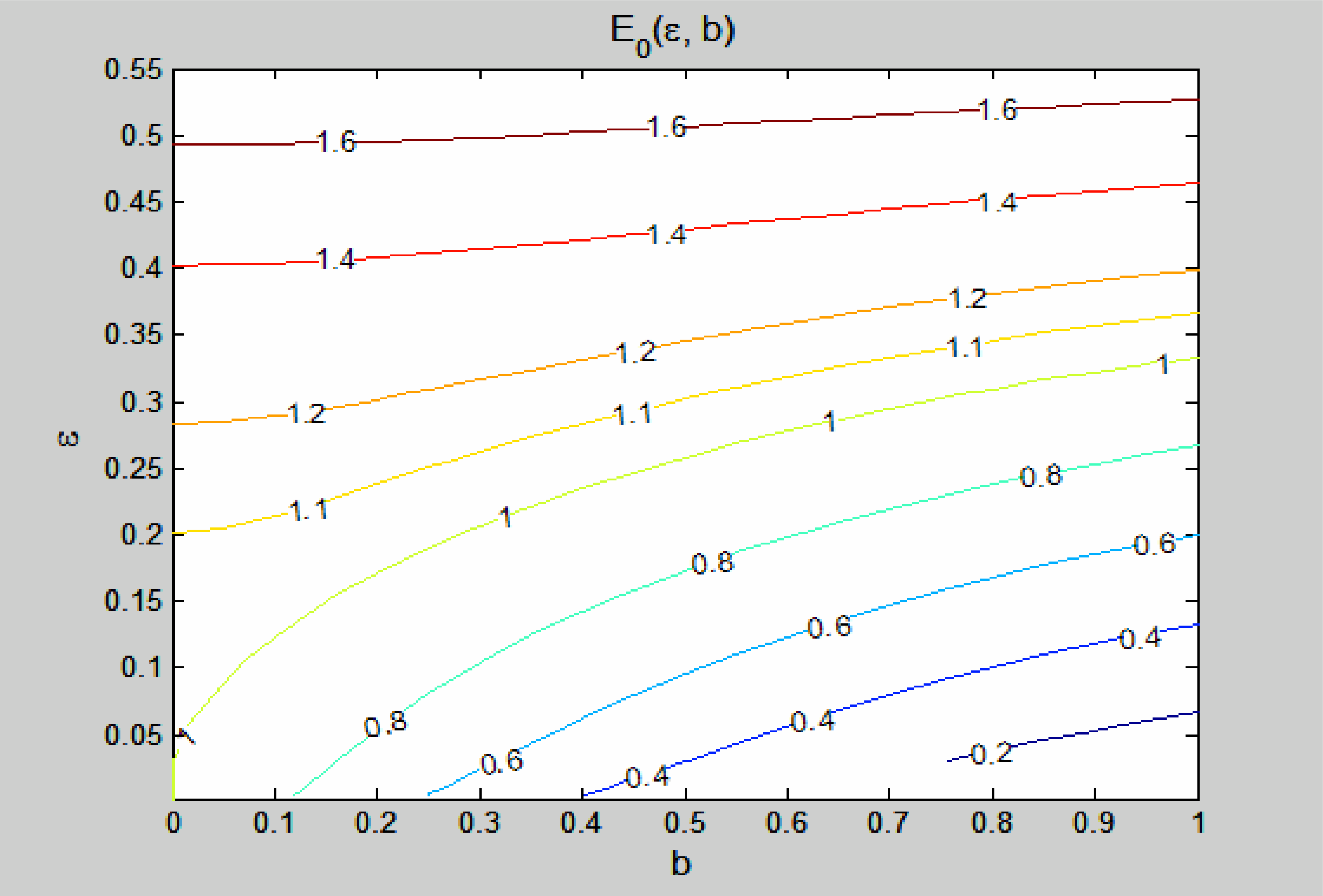
Solving for the low phase speed, C_n, from this expression for E_n (drop the C^2 term from the E(C) cubic relation, $C^2 \ll \alpha$) yields:

$$\boxed{C_n = \frac{-\cos(\phi_0)}{k^2 + E_n \dfrac{\sin^2(\phi_0)}{\alpha}} = \frac{-\cos(\phi_0)}{k^2 + \dfrac{(4n+3)\sin(\phi_0)}{\delta\phi\sqrt{\alpha}}}}$$

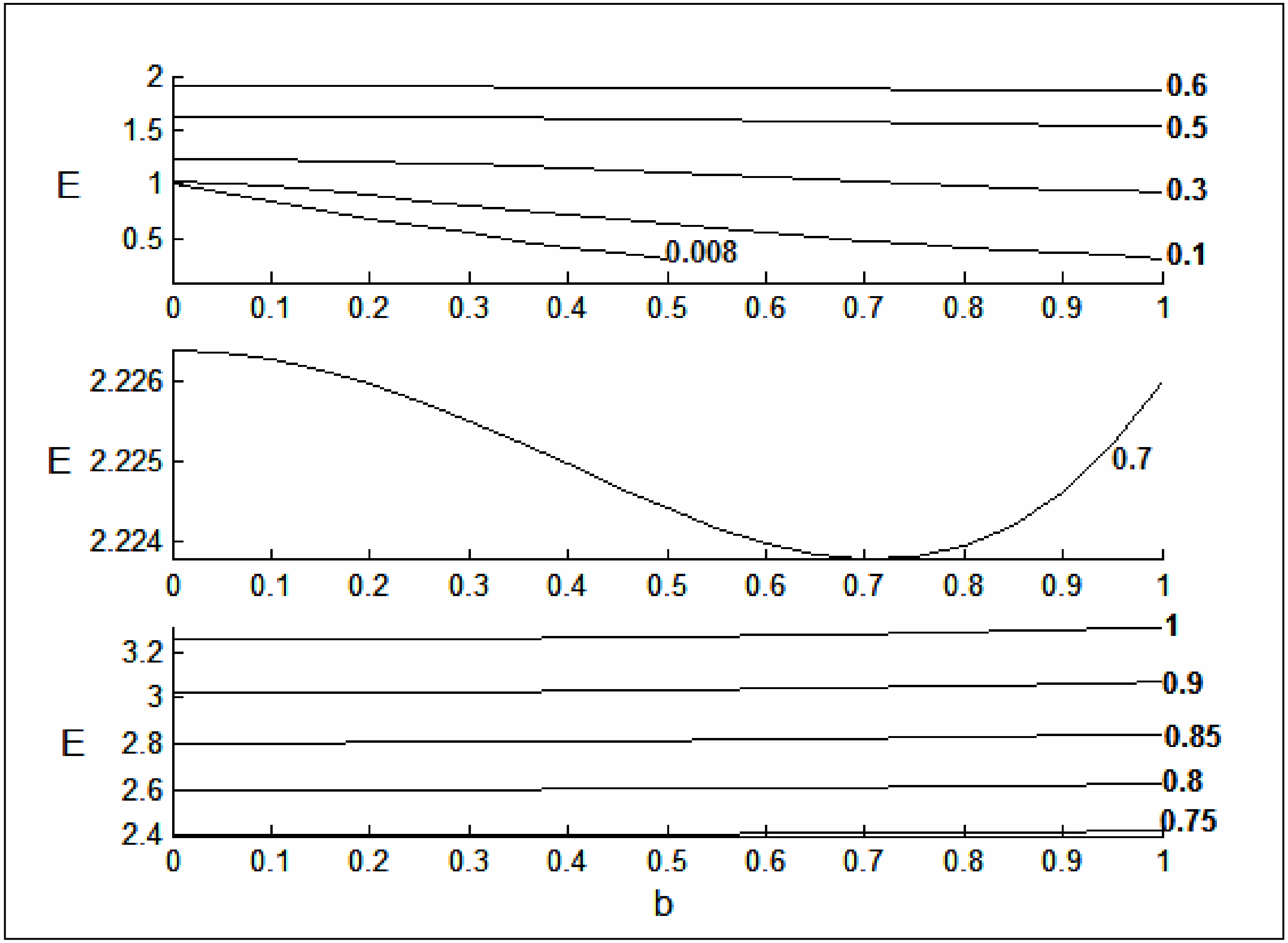
For 0<b<1 the potential of the Schrödinger equation, $(1+bz)^2$, has no simple symmetry on the z=[-1,1] interval or a larger interval



Numerical calculation of the $E_0(\varepsilon, b)$ contours



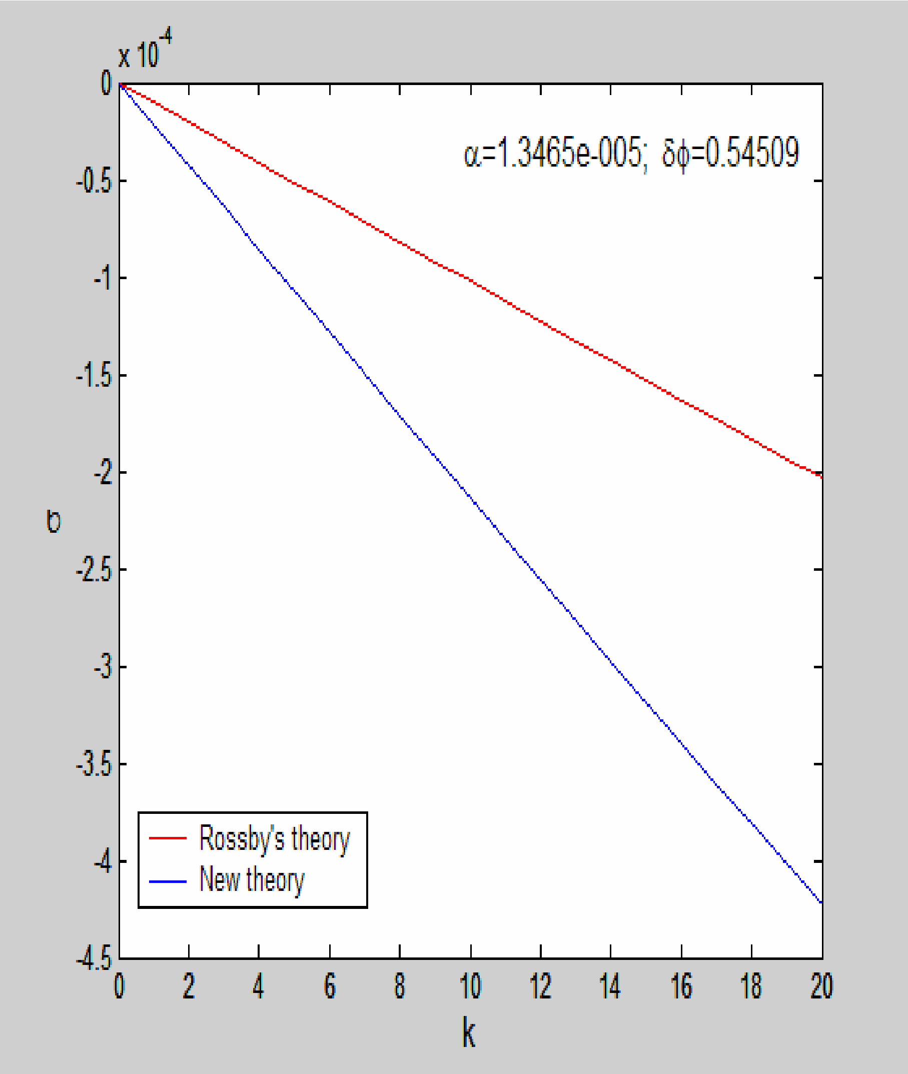
For small ε: E_n decreases with b
For large ε: E_n increases slightly with b

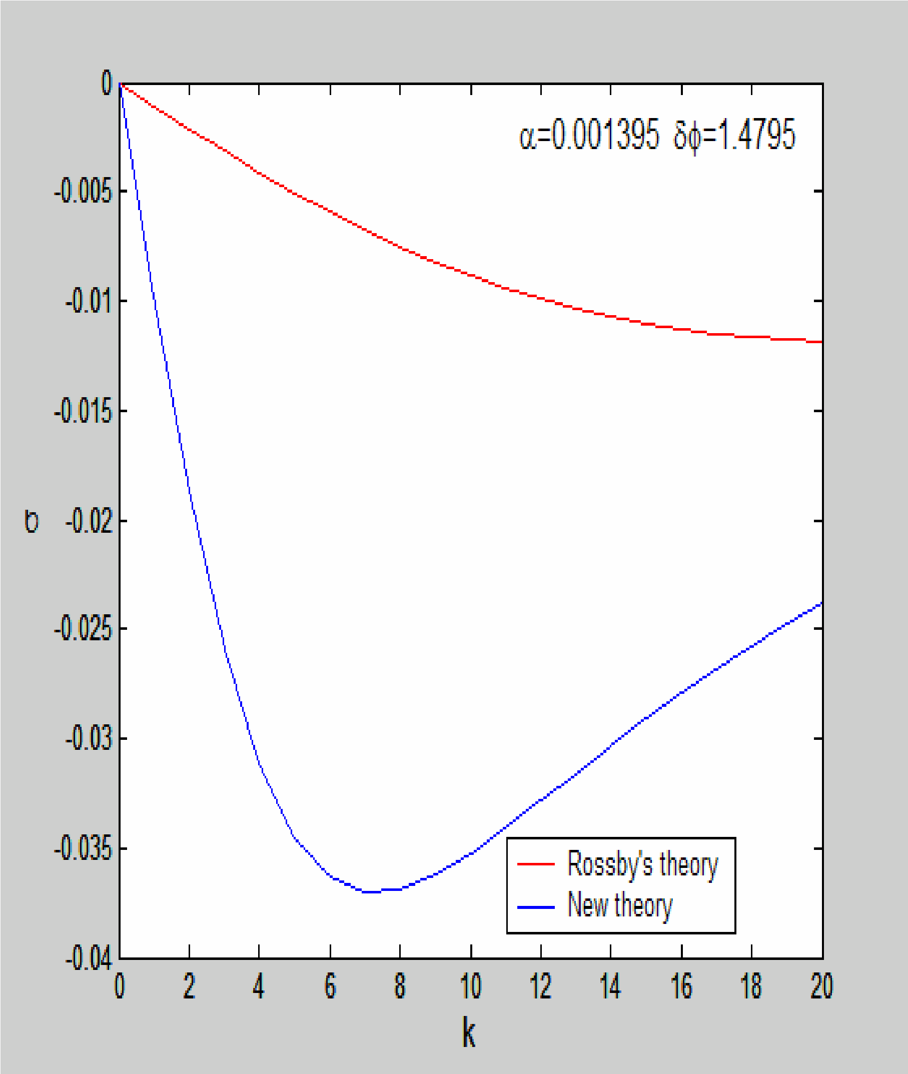


Quantitative Results

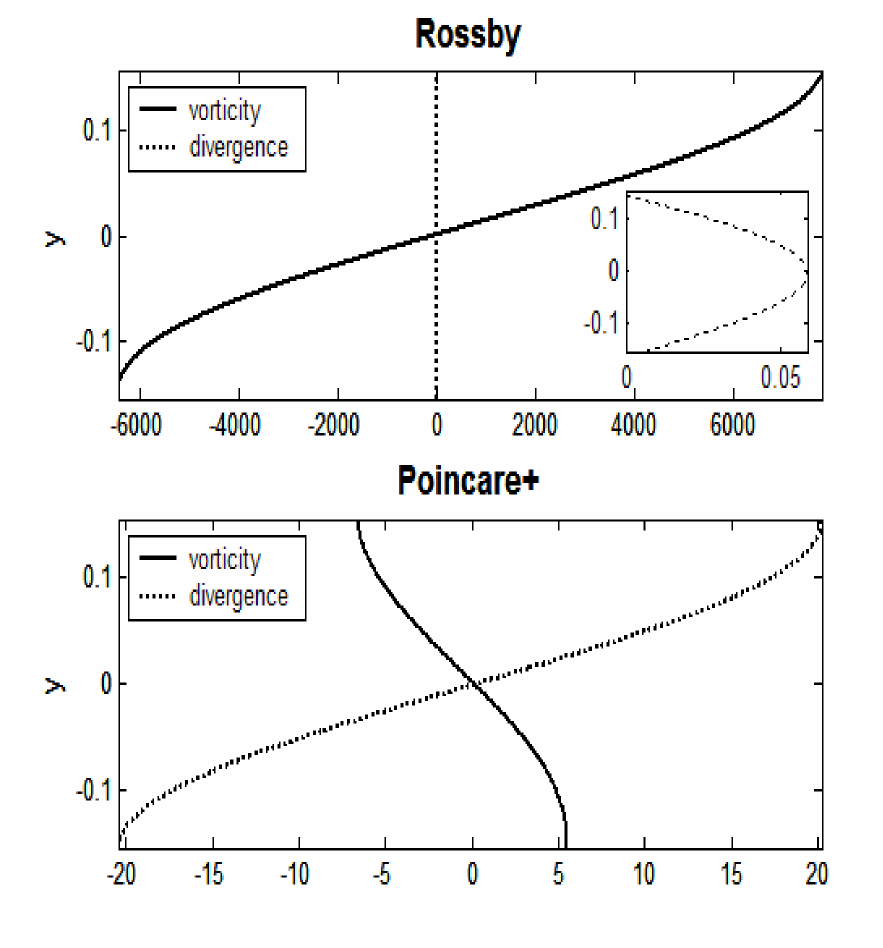
1. For fixed b, $\Delta E_n = E_n(\varepsilon, b) - E_n(\varepsilon, 0)$ increases when $\varepsilon \rightarrow 0$ $\Rightarrow$ error increases when gH get smaller
2. Since E_n decreases with b the phase speed of the corresponding Rossby mode, $C_n \sim -(k^2+[]E_n)^{-1}$, increases (in absolute value) with b $\Rightarrow$ Rossby waves propagate faster in the new, consistent, theory
3. The phase speed of Poincare waves ($C^2 >> \alpha$) increases with E_n ($C^2 = \alpha + E_n \sin^2(\phi_0)/k^2$) i.e. it decreases with b
4. The V(y) eigenfunctions of the Rossby and mode Poincare modes are identical (no f-/β- distinction)
5. The u(y) and η(y) are different for Poincare and Rossby waves only because they are related to V(y) via C

New phase speeds exceed the classical ones by 2-5 for $\alpha \ll 1$ ($\varepsilon \ll 1$) or large $\delta\phi$ (~b)

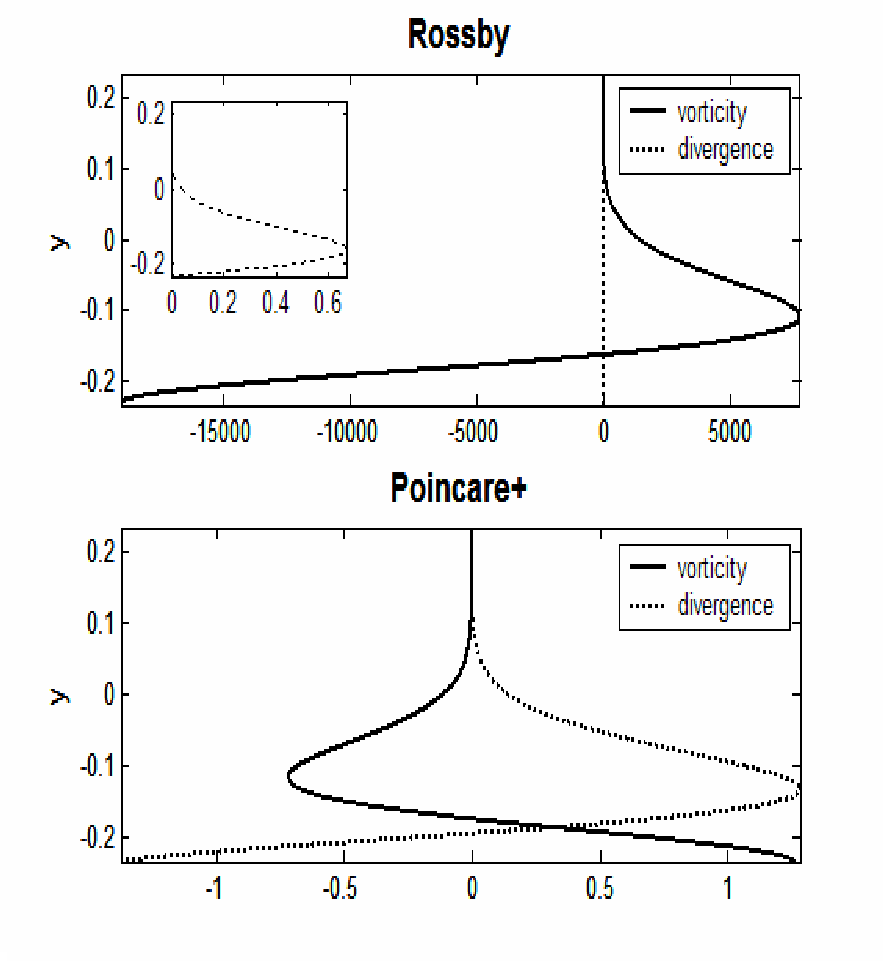




The eigenfunctions' divergence/rotation



Narrow channel; Large α



Wide channel; small α

To sum up:

1. It is possible to solve for both Rossby and Poincare waves in a consistent theory without making the conflicting approximations: $f=f_0$ AND $df/dy=\beta>0$
2. V(y) of Poincare waves is identical to\that of Rossby waves ⇒ Classification to divergent vs. rotational (or f-plane vs. β-plane) modes is a consequence of the different $C(k; E_n)$ relationships
3. Phase speeds of Rossby waves in new theory are 1-4 times LARGER than in Rossby's original theory, as in T/P observations. Difference gets large for **small** g'H'!
4. The new theory extends the fast waves (Poincare and Kelvin) to the β-plane

A different view of waves