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Stable islands in chaotic atom-optics billiards, caused by curved trajectories

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Abstract
We investigate the effects of curving trajectories by applying external force fields on a particle in a billiard. We investigate two special cases: a constant force field and a parabolic potential. These perturbations change the stability conditions and can lead to formation of elliptical orbits in otherwise hyperbolic billiards. We demonstrate these effects experimentally with ultra-cold atoms in atom-optic billiards.

1. Introduction

Billiards are one of the most studied and best understood classes of dynamical systems [1]. The dynamics in billiards can display a broad variety of behaviours, from integrable (e.g. the circular billiard) to chaotic and ergodic (e.g. the Bunimovich stadium or the Sinai billiard [2, 3]), and thereby they demonstrate features of far more complicated systems. Many widely used models of statistical mechanics (e.g. the Lorenz gas) can be reduced to billiards. Of special interest for statistical mechanics are hyperbolic billiards (i.e. billiards with nonvanishing Lyapunov exponent). But what happens if a hyperbolic billiard is perturbed in some way? Will it remain hyperbolic or will elliptical orbits appear? The appearance of elliptical orbits will cause stable islands in phase space, and can greatly affect the transport properties of the systems, for example inducing non-equilibrium states with very long relaxation times, analogues to the Maxwell demon [4, 5]. Chaotic billiards have also been found to be relevant for the design of microlaser cavities [6].

If particles in the billiard move in straight lines between scatterings from the wall, then the movement between scatterings is neutral, and the dynamics is determined solely by the shape of the boundary. Exponential separation of close trajectories (chaotic dynamics) can only occur if the boundary introduces sufficient instability. However, if particles move along curved trajectories between scatterings from the wall the dynamics crucially depend on the specific properties of this motion (we call these billiards curved-trajectory billiards). Since curved-trajectory billiards has been found to be relevant for microdevices [7, 8], much effort has been put into studying them. This includes systems where the curvature of the trajectory arises from a magnetic field [9–12] (trajectories are sections of circles) and from curvature of the surface on which the billiard is made [13]. Quantum/wave effects in curved-trajectory billiards were investigated and scarred wavefunctions were demonstrated [14]. Billiards where gravity provides the confining force along the axis have been studied experimentally, theoretically and numerically [15–19]. Theoretical work on billiards where the curvature is caused by adding a potential to the billiard has been performed [12], where both a constant force field and a parabolic potential were discussed. The stability conditions of one-periodic and symmetric two-periodic orbits were found.

In this work we experimentally investigate classical dynamics of curved-trajectory billiards that would have been hyperbolic if the particle had moved in straight lines. We introduce the curvature by applying external force fields on the particle. We investigate two special cases: particles moving in the presence of (a) a uniform force field (trajectories are sections of parabolas) and (b) a parabolic potential (trajectories are sections of ellipses if the potential is attracting, and sections of hyperbolas if the potential is repulsive). These perturbations can lead to the formation of elliptical orbits in hyperbolic billiards. We experimentally demonstrate these effects, for the first time, using atom-optics billiards [18, 20].

2. Billiards without potential

To determine whether a two-periodic orbit is elliptical, parabolic or hyperbolic, the evolution of a small deviation in initial conditions from the periodic orbit is calculated for one round trip of the periodic orbit to first order in the deviation (the linearized Poincaré map). For a two-dimensional map this yields a 2×2 matrix A, and from its trace the stability of the orbit can be determined: for $\mathrm{tr}^2(A) < 4$ the orbit is elliptical, for $\mathrm{tr}^2(A) = 4$ the orbit is parabolic and for $\mathrm{tr}^2(A) > 4$ the orbit is hyperbolic.

For a two-periodic orbit the trace calculation leads to the well known geometrical stability conditions shown in figure 1 [12], in terms of s, the distance between the two scattering points, and R_1 and R_2, the radii of the corresponding osculating circles (R is positive if the particle scatters from a concave surface). Note that these are the same conditions as geometrical optics gives for the stability of a cavity [21].

3. Gravity

We first see how the presence of a uniform force field (constant acceleration g) effects stability in the billiard. For simplicity we assume a two-periodic orbit along the force field. We assume throughout the paper that the particle possesses enough kinetic energy to access the entire billiard, thus excluding from our discussion gravity-confined billiards such as those described in [15–19], and the one-periodic orbits analysed in [12]. This yields a trace for the linearized Poincaré map given by

$$\mathrm{tr}(A) = 2 - 4\left(\frac{v_2}{R_2} + \frac{v_1}{R_1}\right)t + 4\frac{v_1}{R_1}\frac{v_2}{R_2}t^2 \tag{1}$$

where v_1 and v_2 are the magnitudes of the velocities at encounters with the wall and t is the time between reflections. This trace is very similar to the trace without a force field. There are in general four solutions to the equation $\mathrm{tr}^2(A) = 4$ (just as without the force field), namely $t = 0$, R_2/v_2, R_1/v_1 and $R_2/v_2 + R_1/v_1$. A geometrical interpretation identical to that without the force field is also valid, provided that instead of the geometrical centres of the osculating circles we look at a pair of 'shifted' centres given by $C_i^{sh} = C + \frac{1}{2}gR_i^2/v_i^2$, leading to an effective radius of the osculating circle of $R_i^{eff} = R_i + \frac{1}{2}gR_i^2/v_i^2$. Using the effective radius

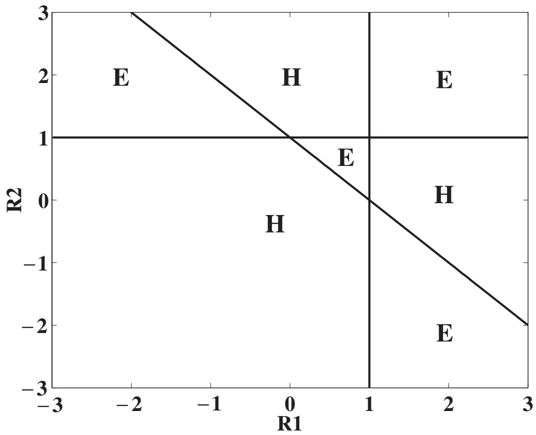

Figure 1. Stability diagram of a two-periodic orbit. s is the distance between the two scattering points; R_1 and R_2 are the radii of the osculating circles at the scattering points (R is positive if the particle scatters from a concave surface). E indicates an elliptical orbit, and H indicates a hyperbolic orbit. The three lines on which the stability changes correspond to (i) the centre of one of the osculating circles being on the opposite boundary of the billiard ($R_1 = s$ or $R_2 = s$) or (ii) the two centres of the two osculating circles coinciding ($R_1 + R_2 = s$).

the stability can be found from figure 1. Note that the shifted centres correspond to points where orbits starting on the osculating circle, and perpendicular to it, will cross.

In the shifted centre picture it is now easy to see when a uniform force field can make stable islands in a hyperbolic billiard of the Bunimovich kind [2]. We demonstrate this experimentally in half a Bunimovich stadium which is obtained by cutting it with a straight wall along its short symmetry axis (see insets in figure 3). This preserves the hyperbolicity of the billiard without a force field. As a force field we use gravity. The half Bunimovich stadium is placed with gravity along its symmetry axis and the arc up. The expression for the gravity-shifted centres indicates that, for atoms with low enough mechanical energy, the centre of the upper arc is shifted to outside the billiard, and thereby the orbit becomes elliptical, and an island of stability appears (the centre of the lower line is always at infinity). Alternatively, if the billiard is turned upside down (the sign of g is changed) the gravity-shifted centre is inside the billiard and it remains hyperbolic. We stress that although the stability of this orbit is determined in the shifted centre picture, the dynamics of the entire billiard is not equivalent to the dynamics of a billiard without a force field constructed according to the shifted centres. Indeed, we shall see new effects such as mixed phase-space and energy-dependent stability, which would not occur in equivalent billiards without a force field. A stability diagram of a half Bunimovich stadium placed in gravity as a function of the energy of the particle can be seen in figure 2, where we see the effect of energy-dependent dynamics: another way of viewing the half Bunimovich stadium in the presence of gravity is as an energy-selective trap/cavity, since it can be designed to support only a narrow energy group in the elliptical island.

Our experimental system for creating atom-optics billiards is described in [20]. Briefly, it is formed using a tightly focused laser beam ($w_0 = 15\ \mu$m), which rapidly (100 kHz) scans the desired billiard shape in the transverse direction by use of two perpendicular acousto-optic scanners (AOSs). The laser beam is tuned 0.5–1.0 nm above the atomic resonance (the D_2 line in ^{85}Rb), hence applying a repulsive dipole force on the atoms. Fast scanning of the beam results in an effective time-averaged potential [22]. Atoms are confined in the longitudinal direction by the beam divergence. Both the mean time for atoms to spontaneously scatter one

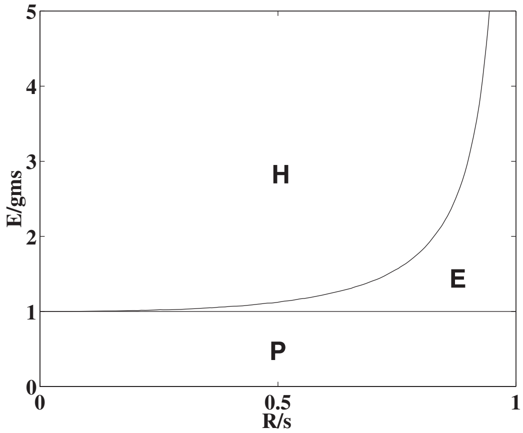

Figure 2. Stability diagram of half a Bunimovich stadium placed in gravity (acceleration g) with the arc up, as a function of the mechanical energy of the particle (E), and radius of the semicircle (R). m is the mass of the particle and s is the distance between scattering points (the height of the billiard). H indicates a hyperbolic orbit, E indicates an elliptic orbit and P indicates a parabolic orbit. The parabolic regime is when the particle does not have enough energy to reach the upper arc, and is therefore just bouncing on the lower straight section. We see that depending on the energy the orbit can be parabolic, elliptic or hyperbolic.

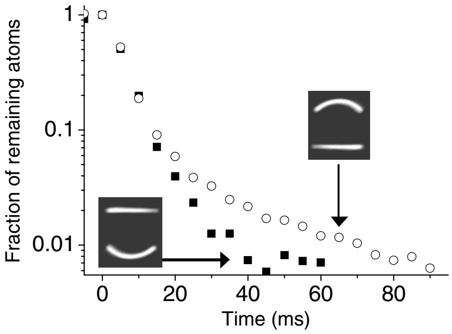

Figure 3. The fraction of remaining atoms as a function of time, measured for half a Bunimovich stadium in a constant force field (gravity). ○: with the arc up. ■: arc down. The existence of a stable island in phase space causes a slowing down in the decay. Inset: measured light distribution of the billiard (205 μm high), with side sections removed.

photon from the laser beam and the mean collision time between trapped atoms are longer than the experiment time, hence the motion of the atoms between reflections from the billiard walls can be regarded as strictly ballistic. Moreover, in spite of several additional complications (such as soft walls, three-dimensional structure, laser beam divergence, imperfections and finite scan rate) the dynamics in such atom-optics billiards is nearly identical to that in ideal two-dimensional 'textbook' billiards of the same shape [20].

Rubidium atoms are loaded into the billiard from a compressed magneto-optical trap and cooled by polarization gradient cooling. A $\sim$1 μs resonant light 'pushing' pulse accelerates the atoms in the radial direction of the trap. A large fraction of the atoms escapes and the remaining $\sim 4 \times 10^5$ atoms have a clipped Gaussian velocity distribution with mean velocity of

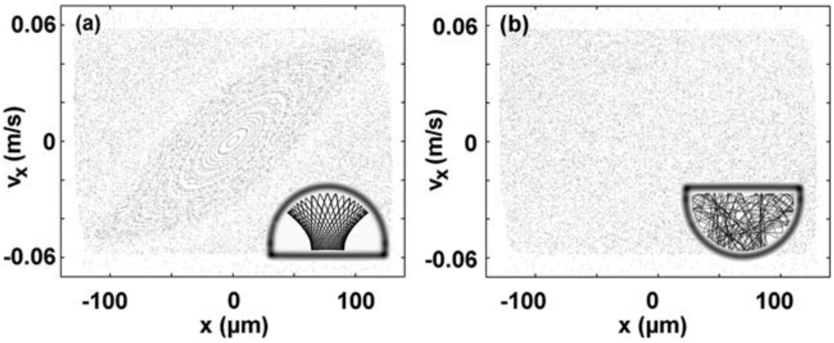

Figure 4. Phase-space diagrams generated by numerical simulations for classical trajectories of Rb atoms inside the half Bunimovich billiard, with the parameters used in the experiment. For clarity, two-dimensional simulations for mono-energetic atoms ($v_2 = 12.3v_{recoil}$) were performed. Phase-space information is presented using a Poincaré surface of a section, showing v_x versus x at every crossing of the $y = 0$ line. (*a*) With arc up: a large island of stability appears around the vertical two-periodic orbit. (*b*) With arc down: completely chaotic phase space. Similar phase-space diagrams with no gravity reveal a similar completely chaotic phase space. Insets show typical trajectories.

$\sim 17v_{recoil}$ and an RMS of $\sim 3v_{recoil}$ ($v_{recoil} \simeq 6$ mm s^{-1})[1]. After the pushing pulse is applied the atoms are left to randomize in phase space for 60 ms. Then (at $t = 0$), sections of the billiard that do not support the island we wish to demonstrate are removed by switching off one of the AOSs synchronously with the scan (see the inset in figure 3). Atoms outside stable regions of phase space can now escape. The fraction of remaining atoms after a certain time is measured by fluorescence detection.

Our half Bunimovich stadium is composed of a semicircle of radius $R_1 = 158$ μm connected to a straight section ($R_2 = \infty$) by two 47 μm straight sections. From figure 2 we see that the velocity regime for which the vertical two-periodic orbit is stable is $10.5v_{recoil} < v_2 < 14v_{recoil}$ for the arc up (velocity measured at the lower straight section). This velocity interval is populated due to the atomic velocity spread. The fraction of remaining atoms is detected both with the arc up and with the arc down (where the two-periodic orbit is always unstable). The results are shown in figure 3. High stability is clearly observed for the arc up in agreement with the above description[2].

To map phase space we performed numerical simulations of the experiment. Figure 4 shows the result of the simulations in the form of a Poincaré surface of a section. As seen in figure 4(*a*) (the arc up), a large stable island is generated by gravity around the orbit predicted to be elliptical. We also performed simulations without gravity, which revealed a completely chaotic phase space, similar to the arc-down picture in figure 4(*b*).

[1] In addition to this velocity distribution we have a small fraction (around 1%) of very slow atoms, which are not accelerated due to imperfections in the pushing beam.

[2] A slowing down of the decay is also seen with the arc down for $t > 40$ ms. This we attribute to $\sim$1% fraction of slow atoms (see footnote 1) that do not reach the upper section of the billiard, and are trapped in the lower arc by the principle described in [15]. This is verified by observing the same fraction of remaining atoms at $t > 40$ ms without the upper straight section. When the same was done with the arc up (leaving only the lower straight section) no stability was detected, as expected.

4. Parabolic potential

We next consider what happens if the atoms move in an attractive harmonic potential, so the trajectories in between scattering from the walls are sections of ellipses. For simplicity we view only what happen to orbits along the radial direction of the potential, and obtain for the trace

$$\mathrm{tr}(A) = 2\left(1 + 2\left(\frac{v_1 v_2}{\omega^2 R_1 R_2} - 1\right)\sin^2 \omega t - \frac{1}{\omega}\left(\frac{v_2}{R_2} + \frac{v_1}{R_1}\right)\sin 2\omega t\right) \quad (2)$$

where ω is the oscillation frequency in the attractive parabolic potential. The stability can again be found from figure 1 using shifted centres or effective radii. The centre of the coordinate system is at the centre of the potential, and the shifted centre will be at $C_i^{sh} = \pm C_i \big/ \sqrt{1 + (\omega R_i/v_i)^2}$, where the minus refers to $R_i < -v_i^2/(\omega^2|x_i|)$ [3]. x_i is the position of the scattering point. From the shifted centres we see that, in contrast to the uniform force field, a two-periodic orbit can become elliptical even if R_1 and R_2 are both negative, since the R corresponding to the shifted centres can be positive even though the geometrical R is negative ($R_i < -v_i^2/(\omega^2|x_i|)$). This means that islands of stability can occur around two-periodic orbits in dispersing billiards and Sinai billiards. It is also easily seen that a potential placed asymmetrically on the long symmetry axis in a Bunimovich stadium can change the stability of the orbit.

We observed islands of stability induced by an attractive parabolic potential experimentally in a dispersing billiard. The billiard is 220 μm high and consists of four convex arcs with the same curvature for each opposite pair (see the inset of figure 5). Two of the arcs have a small radius of $R = -130$ μm and the other two are very weakly curved with a radius of $R = -1$ cm. The parabolic potential is generated by overlapping a standing wave with the billiard (an additional retro-reflected laser beam). The standing wave is red-detuned by 2.5 nm from resonance and has a radial Gaussian profile with dimensions larger than the dimensions of the billiard ($w_0 = 244$ μm), so the potential inside the trap can be approximated with a parabolic potential (the effect of gravity is then simply to shift the centre of the potential). A similar loading procedure to that described above yields a radial Gaussian velocity distribution centred around $16v_{recoil}$ with RMS of $6.5v_{recoil}$. After 60 ms of thermalization time a 205 μm hole was opened on the billiard side and the decay curve from this hole was measured for several powers of the red-detuned laser beam. The results, presented in figure 5, clearly show a slowing down of the decay as the power grows, indicating that an island of stability is formed and grows in size for stronger attractive potentials. When the hole was placed in the bottom of the billiard, we observed a fast decay independent of the strength of the parabolic potential, indicating that the island is indeed formed around a vertical trajectory. It was also verified that no atoms were trapped in the red-detuned standing wave by itself for the powers we used. We performed numerical simulations that confirmed that the island is centred around the two-periodic orbit we predict should become elliptical (figure 6).

An *attractive* potential may intuitively be expected to cause elliptical orbits, since it 'focuses' close trajectories but a *repulsive* potential, for which close orbits separate exponentially in time ($\ddot{x} = Kx$), can also stabilize motion in Bunimovich billiards. The trace of a two-periodic orbit along the radial direction of the potential is quite similar to that for the attractive potential, and assuming $KR_i^2/v_i^2 \leqslant 1$ the shifted centres are located at

[3] Defining the shifted centre of the osculating circle as the point where orbits starting perpendicular to it and displaced an infinitesimal distance along it cross means that every circle has two centres (orbits are ellipses so they cross the x-axis twice). The stability of the orbit can be determined from one of the centres, and the $\pm$ makes sure that it is the correct one that is used.

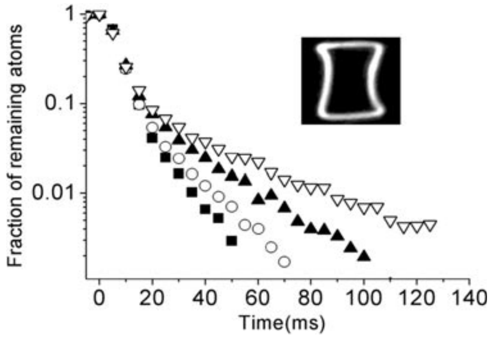

Figure 5. Fraction of remaining atoms in a dispersing billiard as a function of time when a nearly parabolic attractive potential is induced by a standing-wave beam of variable power. ■, 0 mW; ○, 10 mW; ▲, 20 mW; ▽, 30 mW. The formation of a stable island is seen as a slowing down in the decay. Inset: the billiard.

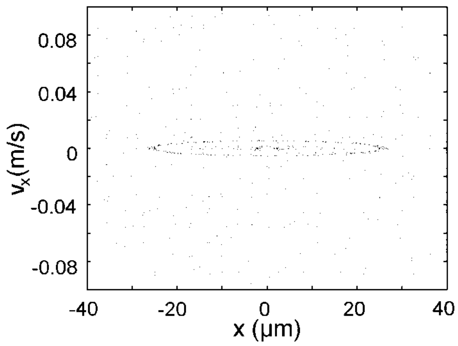

Figure 6. Poincaré surface of sections obtained as in figure 2 for the dispersing billiard with an attractive potential, with parameters similar to the experiment (standing-wave power is 30 mW, atom velocity is $16v_{recoil}$). A large island of stability is seen around the orbit predicted to be elliptical.

$C_i^{sh} = C_i/\sqrt{1 - KR_i^2/v_i^2}$, where the centre of the coordinate system is at the centre of the potential. Using the effective radius corresponding to the shifted centre, the stability is again obtained from figure 1. We see that the centres are repelled by the centre of the potential. From the expression for the shifted centre it is easy to see that a potential placed asymmetrically on the long symmetry axis in a Bunimovich stadium can make the shifted centres cross and thereby change the stability of the orbit. We established formation of islands of stability caused by a repulsive parabolic potential in both analytical calculations of trajectories in the ideal billiard (two dimensional, no gravity, infinite hard walls), and numerical simulations for our experimental billiard, but were not able to produce clear experimental evidence since these islands were small.

5. Summary

In summary, we experimentally confirmed theoretical predictions of stability changes in billiards caused by adding potentials on the billiard. We also demonstrated energy-dependent stability. An example thereof is that an unstable cavity placed in gravity can serve as a velocityselective cavity which is stable only for a narrow velocity group. These results may be applied to the design of atom-optics cavities, to better understand the motion of electrons in quantum dots and to understand properties of billiards in non-Euclidean spaces.

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