ON THE STABILITY OF A SELF-SIMILAR SOLUTION IN THE BURGERS EQUATION

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A new method of investigation of the stability of a solution of a nonlinear equation is suggested, which is based on the isospectral transformation and is applied to the problem of the stability of a self-similar solution in the Burgers model.

1. It is well known that in the investigation of nonlinear wave dynamics fundamental importance is attached to stationary solitary waves, i.e. solitons and scaling solutions. In this respect the problem of their stability is of no less importance. It often occurs that the stability problem leads to the definition of the spectrum of a linear operator. In the present paper a new method based on the solution of the spectral problem for any "simpler" operator is suggested. This "simpler" operator is connected with the primary operator by an isospectral transformation. We use this approach in solving the problem of the stability of a self-similar solution in the Burgers model.

2. Let us suppose that the stability problem is reduced to the definition of the spectrum for some linear operator L:

$$L\psi = \lambda\psi\,, \tag{1}$$

which depends only on one variable x and derivatives with respect to x: $\mathrm{d}/\mathrm{d}x, ..., \mathrm{d}^n/\mathrm{d}x^n$. Let us consider another operator $L_0\,(x, \mathrm{d}/\mathrm{d}x, ..., \mathrm{d}^n/\mathrm{d}x^n) \equiv L_0(x)$ for which the main differential parts of L and L_0 coincide. Then we construct the operator $\mathcal{D}(x,y) = L_0(x) - L_0^{\mathrm{T}}(y)$ (here L^{T} is conjugate to L) and require that the scalar function $F(x,y)$ obeys the equation

$$\mathcal{D}(x,y)F(x,y) = 0\,. \tag{2}$$

Introducing the function $K(x,y)$, connected with $F(x,y)$ by the Marchenko equation

$$K(x,y) + F(x,y) + \int_x^\infty K(x,s)F(s,y)\,\mathrm{d}s = 0 \tag{3}$$

and then applying the operator $\mathcal{D}(x,y)$ to eq. (3) we obtain that $K(x,y)$ satisfies the equation [1,2]

$$\widetilde{\mathcal{D}}(x,y)K(x,y) = [L_1(x) - L_0^{\mathrm{T}}(y)]\,K(x,y) = 0\,, \tag{4}$$

where $L_1(x)$ is expressed through the operators $\mathrm{d}^k/\mathrm{d}x^k$ $(k \leqslant n)$ and derivatives of $K(x,y)$ for $x = y$. It should be noted that the main differential parts of L_1 and L_0 are the same.

Using the Fourier method in (2) and (4) we obtain that the operators L_1 and L_0 have the same spectra. Besides, all eigenfunctions ψ_n of the operator L_1 are connected with eigenfunctions φ_n of L_0 with the same eigenvalue by a triangular representation:

$$\psi_n(x) = \varphi_n(x) + \int_x^\infty K(x,y)\varphi_n(y)\,\mathrm{d}y\,. \tag{5}$$

Thus, if we find the operator L_0 for which $L_1 = L$ then the stability problem (1) reduces to the study of the spectrum of operator L_0.

3. Let us illustrate our method using an example of the Schrödinger operator

$$L_0(x) = \mathrm{d}^2/\mathrm{d}x^2 - U_0(x)$$

for which the eigenfunctions φ_n and the spectrum E_n

are supposed to be known. In this case $F(x,y)$ and $K(x,y)$ obey the following equations:

$$\left[\frac{\partial^2}{\partial x^2} - U_0(x) - \frac{\partial^2}{\partial y^2} + U_0(y)\right] F(x,y) = 0\,, \tag{6}$$

$$\left[\frac{\partial^2}{\partial x^2} - U_0(x) + 2\frac{\mathrm{d}K(x,x)}{\mathrm{d}x} - \frac{\partial^2}{\partial y^2} + U_0(y)\right] K(x,y) = 0\,.$$

The simplest solution of eq. (6) evidently has the form

$$F(x,y) = \sum_n c_n \varphi_n(x)\varphi_n(y)\,.$$

With such a choice of F,

$$K(x,y) = -\sum_n \psi_n(x)\varphi_n(y)\,, \tag{7}$$

where $\psi_n(x)$ are eigenfunctions of the Schrödinger operator with the potential

$$U = U_0(x) - 2\mathrm{d}^2 \ln \det(A_{mn})/\mathrm{d}x^2\,,$$
$$A_{mn} = \delta_{mn} + c_m \int_x^\infty \varphi_n(y)\varphi_m(y)\,\mathrm{d}y\,. \tag{8}$$

It can be shown that the set of eigenfunctions ψ_n is complete, and for all ψ_n [not only for those contained in sum (7)] a triangular representation is valid. It proves that the transformation $U_0 \to U$ is isospectral. It should be noted that only the requirement that matrix A_{mn} be nondegenerate for all x restricts the choice of coefficients c_n.

4. Now we show how this approach can be applied to the Burgers equation,

$$V_t + VV_x - V_{xx} = 0\,, \tag{9}$$

that describes the propagation of weakly nonlinear sound waves in dissipative media.

For this equation it is known [3] that any finite distribution $V(x)$ ($M = \int_{-\infty}^{\infty} V\,\mathrm{d}x < \infty$) evolves to a self-similar solution:

$$V_0(\xi, t) = -2t^{-1/2} f(\xi)\,,$$
$$f(\xi) = \frac{\mathrm{d}}{\mathrm{d}\xi} \ln\left[1 + c_0 \int_\xi^\infty \mathrm{e}^{-\eta^2/4}\,\mathrm{d}\eta\right], \tag{10}$$

where $\xi = xt^{-1/2}$, $c_0 = \pi^{-1/2}\mathrm{e}^{M/4}\sinh(M/4)$. Consider the stability problem of this solution. For modulation of the type

$$\delta V(\xi, t) = \varphi(\xi)\,\psi(\xi)\,\mathrm{e}^{-(1/4+E)\ln t}$$

(where $\varphi = f\mathrm{e}^{\xi^2/8}$) the linearization of eq. (9) on the background of (10) leads to the Schrödinger equation:

$$[-\partial^2/\partial\xi^2 + \tfrac{1}{16}\xi^2 - 2f_\xi - E]\,\psi(\xi) = 0\,. \tag{11}$$

For a perturbation $\delta V(\xi)$ that vanishes for $|\xi| \to \infty$ not slower than $V_0(\xi, t)$ the spectral problem (11) has a spectrum coinciding with that of the oscillator, $E_n = \frac{1}{2}(n + \frac{1}{2})$.

Indeed, for the potential $U_0(\xi) = \frac{1}{16}\xi^2$, $\varphi_n = \mathrm{e}^{\xi^2/8} \times H_n(\frac{1}{2}\xi)$ (here H_n are Hermite polynomials) the simplest addition to the potential [compare with (8)]

$$U'(x) = -2\mathrm{d}K(x,x)/\mathrm{d}x$$
$$= -2\frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln\left[1 + c_0 \int_x^\infty \varphi_0^2(y)\,\mathrm{d}y\right]$$

coincides with $-2f_\xi$ in (11). Therefore such a transformation is an isospectral one, i.e. it retains the spectrum $E_n = \frac{1}{2}(n + \frac{1}{2})$. Then the eigenfunctions of operator (10) can be found with the help of the triangular representation:

$$\psi_n(\xi) = \varphi_n(\xi) + \varphi(\xi) \int_\xi^\infty \varphi_0(\eta)\varphi_n(\eta)\,\mathrm{d}\eta \quad (n \neq 0)\,,$$

$$\psi_0(\xi) = -c_0\varphi_0(\xi)/[1 + c_0 \textstyle\int_\xi^\infty \varphi_0^2(\eta)\,\mathrm{d}\eta] = \varphi(\xi)\,.$$

From this it follows that solution (10) is stable, in particular the modulation $\delta V_0 = \varphi\psi_0 t^{-(1/4+E_0)}$ is neutrally stable and can be obtained as a derivative of the scaling solution (10) with respect to M: $\delta V_0 \sim \partial V_0/\partial M$. It should be emphasized in the above said that for the nonlinear equation $U_t = \hat{S}(U)$ solvable by inverse scattering transform the map $U(x, 0) \to U(x, t)$ for an integrable operator L is also an isospectral one.

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[1] A.B. Shabat, Dokl. Akad. Sci. SSSR, 211 (1973) 1311.
[2] E.A. Kuznetsov and A.V. Mikhailov, Zh. Eksp. Teor. Fiz. 67 (1974) 1717.
[3] V.I. Karpman, Nonlinear waves in dispersive media (Nauka, Moscow, 1973).