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Particle dispersion in a multidimensional random flow with arbitrary temporal correlations

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Abstract

We study the statistics of relative distances $\mathbf{R}(t)$ between fluid particles in a spatially smooth random flow with arbitrary temporal correlations. Using the space dimensionality d as a large parameter we develop an effective description of Lagrangian dispersion. We describe the exponential growth of relative distances $\langle R^2(t)\rangle \propto \exp 2\bar{\lambda}t$ at different values of the ratio between the correlation and turnover times. We find the stretching correlation time which determines the dependence of $\langle \mathbf{R}_1\mathbf{R}_2\rangle$ on the difference $t_1 - t_2$. The calculation of the next cumulant of R^2 shows that statistics of R^2 is nearly Gaussian at small times (as long as $d \gg 1$) and becomes log-normal at large times when large-d approach fails for high-order moments. The crossover time between the regimes is the stretching correlation time which surprisingly appears to depend on the details of the velocity statistics at $t \ll \tau$. We establish the dispersion of the $\ln(R^2)$ in the log-normal statistics. © 1998 Elsevier Science B.V. All rights reserved.

1. Introduction

To describe the statistics of relative distances $\mathbf{R}$ between fluid particles in a turbulent flow is of fundamental interest since it is related to intimate properties of turbulence. The study went in a usual way from gathering experimental data and hypothesizing [1] via phenomenological theories (see Ref. [2] and references therein) to rigorous consideration possible hitherto only under the assumption of a short velocity correlation time [3,4].

Since we want to consider the whole range of correlation times the dimensionality d is the only dimensionless parameter left in our hands for approximations. By considering large d, we make it possible to treat consistently velocity fields with arbitrary

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temporal correlations. There are experimental data suggesting the remarkable persistence of the local straining that determines relative dispersion [5] so that studying a long-correlated case is of particular importance. We derive equations for the correlation functions of inter-particle distances and obtain their solutions that describe both the growth of the mean distance with time and different-time correlations. The main results to be obtained below are likely to be independent of space dimensionality and valid for any correlation time of the velocity field. Our large d approach is very much inspired by the analogous large N methods used already for long time in quantum field theory [6] and statistical mechanics [7]. The most important observation here is that the spatial scalars like $R_a(t_1)R_a(t_2)$ do not fluctuate in the large d limit and do not correlate with each other, they obey closed equations of motion and can be effectively studied in a nonperturbative way.

It is natural to start from considering the dispersion of fluid particles at distances smaller than the viscous scale of turbulence [3–5,8,9]. Velocity field is smooth at such scales, velocity difference is proportional to the distance between points so that the distance increases exponentially with time. We find below the mean stretching rate (Lyapunov exponent) as a function of velocity correlation time and describe different-time correlations of the inter-particle distances. The rate is proportional to the eddy diffusivity (time integral of different-time correlation function) for a short-correlated velocity. For a long-correlated one, the rate is completely determined by simultaneous velocity correlator, i.e. independent of the velocity correlation time; new and somewhat unexpected results are found for the dispersion of the stretching rate in that slow case: the correlation time of the stretching rate grows with the velocity correlation time by the law which is not linear and depends on the behavior of different-time velocity correlation function at small time differences. As the distances outgrow the viscous scale, velocity acquires noninteger scaling exponent $v_R \propto R^{1-\gamma}$ and the exponential growth is replaced by an anomalous (Richardson) diffusion [1]: $\langle R^2(t)\rangle \propto Dt^{2/\gamma}$. Finding the dependence of the diffusivity D on the Lagrangian velocity correlation time will be a subject of future work. Before applying our large dimension approach (which is straightforward) we need to clarify what are possible self-consistent velocity statistics. For example, we show below that velocity cannot be Gaussian with nonzero correlation time and nonzero γ.

Thanks to Kraichnan, we understand that an appropriate way to study scaling is to use Lagrangian variables where masking effect of a large-scale sweeping is absent. Technically, this is a difficult task. We show below that substantial simplifications occur in the limit of large d. This is to be contrasted with the attempt to use infinite d within Eulerian approach where no substantial simplifications have been shown to occur [10]. The simplification of the description of relative dispersion at large d was noticed by Kraichnan in considering the simplest case of a velocity, smooth in space and white in time [3,4]. Large-d approach has been extended for nonsmooth (yet white in time) velocity [11]. Let us present the general formalism for a statistical description of the dispersion of Lagrangian trajectories and introduce the path integral representation of the correlation functions. We will be interested in the dispersion of

Lagrangian trajectories $\rho(t)$ of fluid particles advected by a random velocity field $\mathbf{v}$. The trajectories are defined by the equation $\partial_t\rho = \mathbf{v}(t,\rho)$ and by initial conditions $\rho(0) = \mathbf{r}$ determining positions of fluid particles at $t = 0$. The separation between two particles $\mathbf{R} = \rho_1 - \rho$ starts from its initial value $\mathbf{R}_0 = \mathbf{r}_1 - \mathbf{r}$ and then varies according to $\partial_t\mathbf{R} = \mathbf{v}(t,\rho+\mathbf{R}) - \mathbf{v}(t,\rho)$. We will examine the statistics of the dispersion described by the probability distribution function (PDF) for the quantity $\mathbf{R}(t)$ which has to be determined by averaging over all Lagrangian trajectories. Instead of averaging over functions $\mathbf{R}(t)$ satisfying the equation it is convenient to integrate over all functions $\mathbf{R}(t)$ taking into account the equation as the corresponding functional delta-function. Thus, the integration should be performed with the following weight $\langle\prod_t \delta[\partial_t\mathbf{R} - \mathbf{v}(t,\rho+\mathbf{R}) + \mathbf{v}(t,\rho)]\rangle$, where $\prod_t$ means the product over all moments of time and the angular brackets designate averaging over all Lagrangian trajectories $\rho(t)$. It is convenient to rewrite the product of δ-functions as the path integral over the auxiliary field $\bar{\mathbf{R}}(t)$:

$$\int \mathscr{D}\bar{\mathbf{R}} \exp\left[i\int dt\,(\bar{\mathbf{R}}\partial_t\mathbf{R} - \bar{\mathbf{R}}\{\mathbf{v}(t,\rho+\mathbf{R}) - \mathbf{v}(t,\rho)\})\right] .$$

Then the average of any functional Φ of $\mathbf{R}(t)$ is written as

$$\langle\Phi\rangle = \int \mathscr{D}\mathbf{R}\,\mathscr{D}\bar{\mathbf{R}}\,\exp(iI)\Phi , \tag{1}$$

$$\exp(iI) = \left\langle \exp\left[i\int dt(\bar{\mathbf{R}}\partial_t\mathbf{R} - \bar{\mathbf{R}}\{\mathbf{v}(t,\rho+\mathbf{R}) - \mathbf{v}(t,\rho)\})\right]\right\rangle , \tag{2}$$

where the last averaging $\langle\ldots\rangle$ is performed over the statistics of Lagrangian velocity. We will call I the effective action. Besides the correlation functions of $\mathbf{R}$ it is worth considering also mixed correlation functions of $\mathbf{R}$ and $\bar{\mathbf{R}}$ which have the following meaning. Suppose we add to the right-hand side of the equation for $\mathbf{R}$ a formal external "velocity difference" $\mathbf{V}$ which produces additional stretching $\partial_t\mathbf{R} = \mathbf{v}(t,\rho+\mathbf{R}) - \mathbf{v}(t,\rho)+\mathbf{V}(t)$ and perturbs somehow correlation functions of $\mathbf{R}$. Particularly in the linear approximation the following average arises $\langle R_\alpha(t)\rangle = -i\int dt'\,\langle R_\alpha(t)\bar{R}_\beta(t')\rangle > V_\beta(t')$. Since $\langle R_\alpha(t)\bar{R}_\beta(t')\rangle$ determines the response of $\mathbf{R}(t)$ to $\mathbf{V}(t')$ then $\langle R_\alpha(t)\bar{R}_\beta(t')\rangle = 0$ at $t < t'$ due to causality. More complicated correlation functions with $\bar{\mathbf{R}}$ have analogous meaning of generalized response functions, they possess consequently some causality properties too. Note that the correlation functions $\langle\bar{R}(t_1)\ldots\bar{R}(t_n)\rangle$ containing only $\bar{\mathbf{R}}$ are zero since they determine the response of the unity.

The effective action I in Eq. (1) reflects the statistics of Lagrangian trajectories and, therefore, contains the complete information about correlation functions of the fluid. Generally, it can be written as

$$I = \int dt\,\bar{\mathbf{R}}\partial_t\mathbf{R} + i\int dt_1\,dt_2\,\bar{R}_\alpha(t_1)\mathscr{K}_{\alpha\beta}(t_1-t_2,\mathbf{R}_1,\mathbf{R}_2)\bar{R}_\beta(t_2) + I_{\text{high}} , \tag{3}$$

$$\begin{aligned}\mathscr{K}_{\alpha\beta}(t_1-t_2,\mathbf{R}_1,\mathbf{R}_2) = \tfrac{1}{2}\,\langle(v_\alpha[t_1,\rho(\mathbf{t}_1)+\mathbf{R}_1] - \mathbf{v}_\alpha[\mathbf{t}_1,\rho(\mathbf{t}_1)])\\ \times(v_\beta[t_2,\rho(\mathbf{t}_2)+\mathbf{R}_2] - \mathbf{v}_\beta[\mathbf{t}_2,\rho(\mathbf{t}_2)])\rangle ,\end{aligned} \tag{4}$$

where $I_{\rm high}$ stands for higher order in $\bar{\mathbf{R}}$ terms. For steady turbulence, $\mathscr{K}_{\alpha\beta}$ depends only on the time difference $t_1 - t_2$ and is symmetric under the permutation of $\mathbf{R}_1$ and $\mathbf{R}_2$. Next, $\mathscr{K}_{\alpha\beta}$ tends to zero if $\mathbf{R}_1 \to 0$ or $\mathbf{R}_2 \to 0$. The incompressibility condition $\operatorname{div}\mathbf{v} = 0$ gives

$$\frac{\partial}{\partial R_{1\alpha}}\mathscr{K}_{\alpha\beta}(t_1 - t_2, \mathbf{R}_1, \mathbf{R}_2) = 0\,, \qquad \frac{\partial}{\partial R_{2\beta}}\mathscr{K}_{\alpha\beta}(t_1 - t_2, \mathbf{R}_1, \mathbf{R}_2) = 0\,. \tag{5}$$

Since the correlation function in Eq. (4) is a Lagrangian object in the inertial interval we assume that the velocity correlation time scales as a turnover time. The general scaling arguments gives $\mathscr{K}_{\alpha\beta}(\Lambda^{\gamma}t, \Lambda\mathbf{R}_1, \Lambda\mathbf{R}_2) = \Lambda^{2-2\gamma}\mathscr{K}_{\alpha\beta}(t, \mathbf{R}_1, \mathbf{R}_2)$ with $0 < \gamma < 1$. The correlation time τ_R of Lagrangian motion at the scale R thus behaves as $\tau_R \propto R^{\gamma}$. Kolmogorov's picture corresponds to $\gamma = \frac{2}{3}$.

One cannot say much about correlation functions of $\mathbf{R}$ starting from the general effective action, Eq. (3). It is reasonable to accept some particular form of Eq. (3) and to examine consequences of the expression. The simplest assumption is to neglect the high-order term $I_{\rm high}$ in Eq. (3). Unfortunately, the assumption is not generally self-consistent. The problem is in the intrinsic property of the scheme which requires the symmetry of the action of Eq. (3) with respect to an interchange of the three Lagrangian separations $\mathbf{R}_1, \mathbf{R}_2$ and $\mathbf{R}_1 - \mathbf{R}_2$. In the absence of the high-order term $I_{\rm high}$ that imposes the "triangle identity"

$$\begin{aligned}&\mathscr{K}[t-t', \mathbf{R}_1(t), \mathbf{R}_1(t')] + \mathscr{K}[t-t', \mathbf{R}_2(t), \mathbf{R}_2(t')] - \mathscr{K}[t-t', \mathbf{R}_1(t), \mathbf{R}_2(t')]\\ &\quad - \mathscr{K}[t-t', \mathbf{R}_2(t), \mathbf{R}_1(t')] = \mathscr{K}[t-t', \mathbf{R}_1(t) - \mathbf{R}_2(t), \mathbf{R}_1(t') - \mathbf{R}_2(t')]\,.\end{aligned} \tag{6}$$

The relations are automatically satisfied only in a delta-correlated case (for any γ) or for $\gamma = 0$ (arbitrary temporal correlations). Spatially nonsmooth and temporally correlated velocity cannot be described by the action of the second order in $\bar{\mathbf{R}}$. That shows that the approach usually adopted in studying passive scalar or Lagrangian dispersion, namely "take some (usually Gaussian) velocity statistics and study particle dispersion", does not work for a finite correlated velocity with nonzero γ. In such a general case, different-time velocity statistics has to be restored from single-time statistics. We restrict ourselves by a spatially smooth case where any desirable different-time statistics of a large-scale velocity can be prescribed by external forces. First, this case is of special interest since it is realized at distances smaller than the viscous length. Second, the case allows for a detailed description based on the strain statistics. And finally, it has some peculiarities related to exponential character of stretching. Since the large-scale velocity field is smooth at scales under consideration one can expand $v_\alpha(t, \rho + \mathbf{R}) - v_\alpha(t, \rho) = \sigma_{\alpha\beta}(t)R_\beta$ introducing the local traceless strain tensor $\sigma_{\alpha\beta}$. We thus come to the equation

$$\partial_t R_\alpha(t) = \sigma_{\alpha\beta}(t)R_\beta(t) \tag{7}$$

that allows the statistics of Lagrangian trajectories to be expressed via the statistics of $\sigma_{\alpha\beta}$. Eq. (7) can be rewritten as $\partial_t n_\alpha = \sigma_{\alpha\beta}n_\beta - \sigma_{\gamma\beta}n_\beta n_\gamma n_\alpha$ and $\partial_t \ln R = \sigma_{\gamma\beta}n_\beta n_\gamma$, where $n_\alpha = R_\alpha/R$. Since $\sigma_{\gamma\beta}n_\beta n_\gamma$ is a random function independent of R the central limit theorem leads to the conclusion that the stretching rate $\ln[R(t)]/t$ has a normal distribution at $t \to \infty$

$$\mathscr{P}(R) \propto \exp\{-[\ln(R/R_*) - \bar{\lambda}t]^2/(2\Delta t)\}\,, \tag{8}$$

where $\bar{\lambda}$ is the mean stretching rate and Δ is a dispersion. Statistics of R is thus asymptotically log-normal at $t \gg \tau$ where τ is the strain Lagrangian correlation time. According to Eq. (8) one has to have

$$\langle R^{2n}\rangle = R_*^{2n}\exp(2n\bar{\lambda}t + 2n^2\Delta t)\,. \tag{9}$$

The most general expression for the strain correlation function is

$$\langle\sigma_{\alpha\mu}(t)\sigma_{\beta\nu}(0)\rangle = \{g[\delta_{\alpha\beta}\delta_{\mu\nu} - d^{-1}\delta_{\alpha\mu}\delta_{\beta\nu}] + d^{-1}g_1[\delta_{\alpha\beta}\delta_{\mu\nu} - \delta_{\alpha\nu}\delta_{\beta\mu}]\}D/(d\tau)\,,$$

where D is the strain amplitude and $g(t/\tau)$, $g_1(t/\tau)$ are some integrable dimensionless functions. The normalization condition: $\int d\xi\, g(\xi) = 1$. One can also write $\langle\sigma(t)\sigma(0)\rangle$ in terms of the correlation function of Eq. (4) which is now bilinear in the separations. If $t = 0$ then Eq. (4) can be obviously rewritten in terms of the simultaneous pair correlation function of the velocity. That leads to the relation $g_1(0) = g(0)$ which justifies the normalization. Generally, the statistics of the strain $\sigma_{\alpha\beta}$ is non-Gaussian. Therefore, high-order cumulants of $\sigma_{\alpha\beta}$ should be taken into account as well. Below we will treat the Gaussian statistics of the strain discussing only shortly the effects related to high-order cumulants. Rewriting Eq. (3) for the case under consideration we get

$$\begin{aligned} iI = i\int d\zeta\,\bar{\mathbf{R}}\partial_\zeta\mathbf{R} - \frac{\beta}{2d}\int d\zeta\, d\zeta'\, g(\zeta-\zeta')\left\{(\bar{\mathbf{R}}\bar{\mathbf{R}}')(\mathbf{R}\mathbf{R}') - \frac{(\bar{\mathbf{R}}\mathbf{R}')(\bar{\mathbf{R}}'\mathbf{R}')}{d}\right\} \\ -\frac{\beta}{2d^2}\int d\zeta\, d\zeta'\, g_1(\zeta-\zeta')\{(\bar{\mathbf{R}}\bar{\mathbf{R}}')(\mathbf{R}\mathbf{R}') - (\bar{\mathbf{R}}\mathbf{R}')(\bar{\mathbf{R}}'\mathbf{R})\} + iI_{\mathrm{high}}\,. \end{aligned} \tag{10}$$

Here $\zeta = t/\tau$ is the dimensionless time and $\beta = D\tau$. The term I_{high} designates contributions to the action originating from the high-order cumulants of $\sigma_{\alpha\beta}$.

2. Large dimensionality

It is hard to calculate correlation functions of $\mathbf{R}$ even with the effective action, Eq. (10). An essential progress can be achieved in the limit of large d where spatial scalars like $R_\alpha(t_1)R_\alpha(t_2)$ do not fluctuate because of the self-averaging phenomenon: the individual terms in the sum over α fluctuate strongly while the sum as a whole does not. Particularly in large-d limit, R^2 does not fluctuate and the average $\langle R^2(t)\rangle$ completely characterizes the statistics of R^2: e.g. $\langle R^4(t)\rangle = \langle R^2(t)\rangle^2$. Eq. (9) implies that the large-d approach does not work at large enough t when $\langle R^4(t)\rangle/\langle R^2(t)\rangle^2 = \exp(4\Delta t) \gg 1$. We show below that $\Delta \propto 1/d$ which provides for a long time interval when $1/d$ approach is still valid and PDF of Eq. (8) is already realized. In this interval,

$\tau \ll t \ll \Delta^{-1}$, moments of R^2 can be found by large-d approach and compared with Eq. (9). Such a comparison gives $\bar{\lambda}$ and Δ in Eq. (8). We sketch out a systematic $1/d$ perturbation theory based on the Schwinger–Dyson relations in the presence of the sources generating the correlation functions of the spatial scalars. Let us add to Eq. (10) external sources $X(\zeta_1,\zeta_2), Y(\zeta_1,\zeta_2), Z(\zeta_1,\zeta_2)$ bilocal in time:

$$iI_{xyz} = iI + \int d\zeta_1\, d\zeta_2\, [X(\bar{\mathbf{R}}_1\mathbf{R}_2) + Y(\mathbf{R}_1\mathbf{R}_2) + Z(\bar{\mathbf{R}}_1\bar{\mathbf{R}}_2)]\,, \tag{11}$$

so that the partition function $\mathscr{Z}_{xyz} = \int \mathscr{D}\mathbf{R}\, \mathscr{D}\bar{\mathbf{R}} \exp(iI_{xyz})$ contains the complete information about correlation functions of $\mathbf{R}$, $\bar{\mathbf{R}}$. For example,

$$F(\zeta_1,\zeta_2) = \frac{1}{d}\,\langle R_\alpha(\zeta_1)R_\alpha(\zeta_2)\rangle = \frac{1}{d}\,\frac{\delta \log \mathscr{Z}_{xyz}}{\delta Y(\zeta_1,\zeta_2)}\bigg|_0\,, \tag{12}$$

$$G(\zeta_1,\zeta_2) = \frac{1}{d}\langle R_\alpha(\zeta_1)\bar{R}_\alpha(\zeta_2)\rangle = \frac{1}{d}\frac{\delta \log \mathscr{Z}_{xyz}}{\delta X(\zeta_1,\zeta_2)}\bigg|_0\,, \tag{13}$$

where $\ldots|_0$ means that zero values of X, Y, Z should be substituted. The angular brackets in Eqs. (12) and (13) designate averaging Eq. (1). Remind that due to causality $G(\zeta_1,\zeta_2)=0$ if $\zeta_1 < \zeta_2$ and $\langle \bar{R}_\alpha(\zeta_1)\bar{R}_\alpha(\zeta_2)\rangle = 0$. To develop a compact $1/d$ perturbation theory, it is convenient to introduce more general objects

$$\mathscr{F} = \frac{1}{d}\frac{\delta \log \mathscr{Z}_{xyz}}{\delta Y(\zeta_1,\zeta_2)}\,, \quad \mathscr{G} = \frac{1}{d}\frac{\delta \log \mathscr{Z}_{xyz}}{\delta X(\zeta_1,\zeta_2)}\,, \quad \bar{\mathscr{F}} = \frac{1}{d}\frac{\delta \log \mathscr{Z}_{xyz}}{\delta Z(\zeta_1,\zeta_2)}\,, \tag{14}$$

the last correlation function is nonzero since the "sources" break causality. The objects introduced enable one to obtain multi-point correlation functions. For example, the four-point correlation function of $\mathbf{R}$ can be expressed as

$$\frac{1}{d^2}\,\langle R_\alpha(\zeta_1)R_\alpha(\zeta_2)R_\beta(\zeta_3)R_\beta(\zeta_4)\rangle = \frac{1}{d}\frac{\delta \mathscr{F}(\zeta_1,\zeta_2)}{\delta Y(\zeta_3,\zeta_4)}\bigg|_0 + F(\zeta_1,\zeta_2)F(\zeta_1,\zeta_2)\,. \tag{15}$$

Using the relation $\int \mathscr{D}\mathbf{R}\, \mathscr{D}\bar{\mathbf{R}}\, \delta[R_\alpha(\zeta')\exp(iI_{xyz})]/\delta\bar{R}_\alpha(\zeta) = 0$ and analogs for other combinations of $\mathbf{R}$, $\bar{\mathbf{R}}$, we can get Schwinger–Dyson equations for Eq. (14). For example, from Eq. (10) we get omitting terms originating from I_{high}

$$\begin{aligned}
& i\partial_{\zeta_1}\mathscr{F}(\zeta,\zeta_1) + \int d\zeta_2 [X(\zeta_1,\zeta_2)\mathscr{F}(\zeta,\zeta_2) + 2Z(\zeta_1,\zeta_2)\mathscr{G}(\zeta_2,\zeta)] \\
& -\frac{\beta}{d}\int d\zeta_2\, g(\zeta-\zeta_2)\left[\mathscr{F}(\zeta,\zeta_1)\bar{\mathscr{F}}(\zeta_2,\zeta_2) + \frac{1}{d}\frac{\delta\mathscr{F}(\zeta,\zeta_1)}{\delta Z(\zeta_2,\zeta_2)}\right] \\
& +\frac{\beta}{d^2}\int d\zeta_2\, g_1(\zeta-\zeta_2)\left[\mathscr{F}(\zeta,\zeta_2)\bar{\mathscr{F}}(\zeta_1,\zeta_2) + \frac{1}{d}\frac{\delta\mathscr{F}(\zeta,\zeta_2)}{\delta Z(\zeta_1,\zeta_2)}\right] \\
& -\beta\int d\zeta_2\left[g(\zeta-\zeta_2) + \frac{g_1(\zeta-\zeta_2)}{d}\right]\left[\mathscr{G}(\zeta_2,\zeta)\mathscr{F}(\zeta_1,\zeta_2) + \frac{\delta\mathscr{F}(\zeta_1,\zeta_2)}{d\,\delta X(\zeta_2,\zeta_1)}\right] = 0\,.
\end{aligned} \tag{16}$$

One can obtain one similar equation for the symmetric $\bar{\mathscr{F}}$ and two for nonsymmetric $\mathscr{G}$. Starting from Eq. (16) one formulates the regular $1/d$ expansion for the correlation functions of $\mathbf{R}$ and $\bar{\mathbf{R}}$. In the main approximation, dropping all the explicit $1/d$-terms we get at $X = Y = Z = 0$ (due to causality $\bar{\mathscr{F}}|_0 = 0$)

$$-i\partial_\zeta G(\zeta,\zeta') = \delta(\zeta-\zeta')\,, \qquad i\partial_\zeta G(\zeta',\zeta) = \delta(\zeta-\zeta')\,, \tag{17}$$

$$i\partial_{\zeta_1} F(\zeta_1,\zeta_2) = \beta\int_0^\infty d\zeta_3\, G(\zeta_2,\zeta_3) g(\zeta_1-\zeta_3) F_{\zeta_1\zeta_3}\,. \tag{18}$$

To find the contribution in the next order over $1/d$ to, say, $F(\zeta_1,\zeta_2)$, we write $F(\zeta_1,\zeta_2,) = F_0(\zeta_1,\zeta_2,) + F_1(\zeta_1,\zeta_2,)/d + O(1/d^2)$ and similarly for G where F_0, G_0 obey the $d\to\infty$ equations, Eq. (18), and then plug them into the relations like Eq. (16). Then we keep all the terms of the order $1/d$ (the zero-order terms cancel by virtue of the $d\to\infty$ equations) and put $X = Y = Z = 0$. This gives

$$i\partial_{\zeta_1} F_1(\zeta,\zeta_1) - \beta\int d\zeta_2\, g(\zeta-\zeta_2)\Big[G_0(\zeta_2,\zeta)F_1(\zeta_1,\zeta_2) + G_1(\zeta_2,\zeta)F_0(\zeta_1,\zeta_2) + \frac{\delta\mathscr{F}_0(\zeta_1,\zeta_2)}{\delta X(\zeta_2,\zeta)}\Big|_0\Big] - \beta\int d\zeta_2\, g_1(\zeta-\zeta_2)G_0(\zeta_2,\zeta)F_0(\zeta_1,\zeta_2) = 0\,. \tag{19}$$

Functional derivative here can be found from the equation obtained by varying Eq. (16) with respect to X:

$$i\partial_{\zeta_1}\frac{\delta\mathscr{F}_0(\zeta,\zeta_1)}{\delta X(\zeta_3,\zeta_4)}\Big|_0 - \beta\int d\zeta_2\, g(\zeta-\zeta_2)\Big[G_0(\zeta_2,\zeta)\frac{\delta\mathscr{F}_0(\zeta_1,\zeta_2)}{\delta X(\zeta_3,\zeta_4)}\Big|_0 + F_0(\zeta_1,\zeta_2)\frac{\delta\mathscr{G}_0(\zeta_2,\zeta)}{\delta X(\zeta_3,\zeta_4)}\Big|_0\Big] + \delta(\zeta_1-\zeta_3)F(\zeta,\zeta_4) = 0. \tag{20}$$

Other functional derivatives and the corrections to G and $\bar{F}$ can be obtained similarly. Equations of the type Eqs. (19) and (20) define the first $1/d$ corrections to the two-point correlators. We will be interested mainly in the quantities

$$F(\zeta,\zeta',\zeta'') = \frac{\delta\mathscr{F}(\zeta,\zeta')}{\delta Y(\zeta'',\zeta'')}\Big|_0\,, \qquad G(\zeta,\zeta',\zeta'') = \frac{\delta\mathscr{G}(\zeta,\zeta')}{\delta Y(\zeta'',\zeta'')}\Big|_0\,, \tag{21}$$

$$\bar{F}(\zeta,\zeta',\zeta'') = \frac{\delta\bar{\mathscr{F}}(\zeta,\zeta')}{\delta Y(\zeta'',\zeta'')}\Big|_0 = \frac{1}{d}\left\langle \bar{R}_\alpha(\zeta)\bar{R}_\alpha(\zeta')R^2(\zeta'')\right\rangle\,, \tag{22}$$

describing finite d corrections to the statistics of $R^2(t)$. The $1/d$-terms in Eq. (10) lead to contributions to Eqs. (21) and (22) which are $1/d$ times the respective reducible expressions, they do not produce any qualitative effect and will be omitted below. We use the effective action $iI = i\int d\zeta\, \bar{\mathbf{R}}\partial_\zeta\mathbf{R} - \beta\int d\zeta\, d\zeta'\, g(\zeta-\zeta')(\bar{\mathbf{R}}\bar{\mathbf{R}}')(\mathbf{R}\mathbf{R}')/2d$, obtained from Eq. (10) in the main order and get for $G(\zeta,\zeta',\zeta'')$

$$-i\partial_\zeta G + \beta\int d\zeta_2 g(\zeta-\zeta_2)\bar{F}(\zeta',\zeta_2,\zeta'')F(\zeta,\zeta_2) = 0\,, \tag{23}$$

$$i\partial_{\zeta'} G + \beta\int d\zeta_2 g(\zeta'-\zeta_2)\bar{F}(\zeta',\zeta_2,\zeta'')F(\zeta,\zeta_2) = 2\delta(\zeta'-\zeta'')F(\zeta',\zeta)\,, \tag{24}$$

$$[\partial_\zeta \partial_{\zeta_1} - \beta g(\zeta - \zeta_1)]F(\zeta, \zeta_1, \zeta''),$$
$$= -\beta^2 \int d\zeta_2 F(\zeta, \zeta_2) g(\zeta - \zeta_2) \int d\zeta_3 g(\zeta_1 - \zeta_3) F(\zeta_1, \zeta_3) \bar{F}(\zeta_2, \zeta_3, \zeta''), \tag{25}$$

$$[\partial_\zeta \partial_{\zeta_1} - \beta g(\zeta - \zeta_1)]\bar{F}(\zeta_1, \zeta, \zeta'') = -2\delta(\zeta - \zeta'')\delta(\zeta - \zeta_1). \tag{26}$$

3. Pair correlation functions and cumulants

We are interested mainly in the statistics of $\langle R^2(\zeta)\rangle$ which does not satisfy any closed equation so we first find Eqs. (12) and (21) and then extract cumulants $\langle R^2(\zeta)\rangle = dF(\zeta,\zeta)$ and $\langle R^2(\zeta)R^2(\zeta')\rangle - \langle R^2(\zeta)\rangle\langle R^2(\zeta')\rangle = dF(\zeta,\zeta,\zeta')$. To calculate Eqs. (12) and (21) one should analyze Eqs. (24) and (26) besides Eqs. (17) and (18). Let us stress that Eqs. (12) and (21) describe the correlation of Lagrangian distances at different instants. This correlation cannot be obtained using dimensional arguments. Eq. (17) should be solved taking into account causality: $G(\zeta,\zeta') = 0$ if $\zeta < \zeta'$. Such retarded solution of Eq. (17) is simply $G(\zeta,\zeta') = i\theta(\zeta-\zeta')$, where θ is the step function. Taking the derivative of Eq. (18) over ζ_2 and using Eq. (17) we obtain

$$\frac{\partial^2}{\partial\zeta_1\partial\zeta_2} F(\zeta_1,\zeta_2) = \beta g(\zeta_1 - \zeta_2) F(\zeta_1,\zeta_2). \tag{27}$$

Because of the condition $G(\zeta_2,\zeta_3) = 0$ if $\zeta_2 < \zeta_3$, we get from Eq. (18) the initial condition $\partial_\zeta F(\zeta,0) = 0$. Since Eq. (27) is linear and the coefficient at the right-hand side explicitly depends only on the difference $\zeta_1 - \zeta_2$ a solution of Eq. (27) can be written as a sum of harmonics $F(\zeta_1,\zeta_2) = \exp[\lambda(\zeta_1+\zeta_2)/2]\Psi(\zeta_1-\zeta_2)$. Since $F(\zeta_1,\zeta_2) = F(\zeta_2,\zeta_1)$ then $\Psi(t)$ is an even function which satisfies a Schrödinger equation with the strain correlation function as a potential

$$(\lambda^2/4 - \partial_\zeta^2)\Psi(\zeta) = \beta g(\zeta)\Psi(\zeta). \tag{28}$$

There exists the spectrum of λ. Yet at large time, only the largest exponent λ_0 contributes, which corresponds to the ground state. The relation between the stretching rate λ_0 (related to the energy of the ground state) and the parameter β (which determines the depth of the well) can be readily established. For a shallow well $\lambda_0^2 \propto \beta^2$ while for a deep well the lowest level $\lambda_0^2 \propto \beta$. Comparing the behavior of $F(\zeta,\zeta) = d^{-1}\langle R^2\rangle$ with Eq. (9) we conclude that $\bar{\lambda} \sim D$ in the fast case and $\bar{\lambda} \sim \sqrt{D/\tau}$ in the slow case. The same behavior of $\bar{\lambda}(\beta)$ has been discovered before for the two-dimensional case [9]. Considering $\zeta_1 - \zeta_2 \gg 1$, we can neglect the right-hand side in Eq. (28) and see that the same stretching rate λ determines both the growth of $\langle R^2(\zeta)\rangle$ and the decay of different-time correlation at $\zeta_1 + \zeta_2$ fixed. That also shows that the correlation function is becoming independent of the larger time when time difference exceeds τ.

For the fast strain corresponding to $\beta \ll 1$ one can put $g(\zeta) \to 2\delta(\zeta)$, then Eq. (28) has unique and amazingly simple solution corresponding to $\lambda = 2\beta$: $F(\zeta_1,\zeta_2) = R_0^2\exp[\beta(\zeta_1+\zeta_2-|\zeta_1-\zeta_2|)] = R_0^2\exp[2\beta\min(\zeta_1,\zeta_2)]$. Since $\bar{F}(\zeta,\zeta',\zeta'')$ is a susceptibility of $\langle R^2(\zeta'')\rangle$ with respect to the strain at ζ and ζ' we conclude that $\bar{F} = 0$ at $\zeta'' <$

$\max(\zeta,\zeta')$. Solving Eq. (26) and Eq. (25), one obtains $\bar F(\zeta,\zeta',\zeta'') = -2\exp[2\beta\zeta'' - 2\beta\max(\zeta,\zeta')]\theta(\zeta''-\zeta)\theta(\zeta''-\zeta')$ and $F(\zeta,\zeta_1,\zeta'') = 4\beta F(\zeta,\zeta_1)F(\zeta'',\zeta'')[\min\{\zeta'', \max(\zeta,\zeta_1)\} + \min(\zeta,\zeta_1,\zeta'')]$ so that

$$\frac{\langle\langle R^2(\zeta)R^2(\zeta'')\rangle\rangle}{\langle R^2(\zeta)\rangle\,\langle R^2(\zeta'')\rangle} = \frac{F(\zeta,\zeta,\zeta'')}{dF(\zeta,\zeta)F(\zeta'',\zeta'')} = \frac{8\beta}{d}\min\{\zeta'',\zeta\}\,, \tag{29}$$

which describes the growth of the cumulant with respect to the reducible part. At large times the cumulant of R^2 outgrows the reducible correlator. This corresponds to the log-normality of $R^2(t)$ at large t where the $1/d$ approach is violated. Let us compare Eq. (29) with Eq. (9) valid at $t \gg \tau$ i.e. at $\zeta \gg 1$. It follows from Eq. (29) and Eq. (9) that while large-d approach for $\langle R^2(t)\rangle$ is valid uniformly in time, the calculation of higher moments is restricted in time. Eq. (29) works if $\langle R^4\rangle - \langle R^2\rangle^2 \ll \langle R^2\rangle^2$, which is satisfied for Eq. (9) if $\Delta t \ll 1$. Thus the comparison performed at $t/\tau \ll \Delta t \ll 1$ gives $\bar\lambda = \lambda_0/(2\tau) = \beta/\tau = D$. Regarding $\Delta t \ll 1$ we get from Eq. (9), $[\langle R^4(t)\rangle - \langle R^2(t)\rangle^2]/\langle R^2(t)\rangle^2 \approx 8\Delta t$. Comparing that with Eq. (29) we get $\Delta = \bar\lambda/d$ which agrees with the large-d limit of Δ obtained in Ref. [12].

Let us turn to the slow case $\beta \gg 1$. The stretching rate at a given value of D/τ (determining the simultaneous correlation function of the strain) is independent of τ in this case. Here we establish the correlation time of the stretching rate and its dispersion. The correlation time, related to the dependence of F on relative time, can be found from the Schrödinger equation, Eq. (28). At large β the potential in Eq. (28) is deep. The behavior of F at large ζ_1,ζ_2 is associated with the lowest levels determined by the function $g(\zeta)$ at small ζ. Assuming a power dependence $g(0) - g(\zeta) \propto |\zeta|^a$, we obtain the expression $-\beta_2 + \beta_1|\zeta|^a$ for the potential $-\beta g(\zeta)$. All answers can be expressed via the normalized eigenfunctions of the Schrödinger equation $[-\partial_\zeta^2 + \beta_1|\zeta|^a]\Psi_n(\zeta) = s_n\Psi_n(\zeta)$, $\int d\zeta\,|\Psi_n|^2 = 1$. For $a = 2$ the functions Ψ_n are expressed via Hermit polynomials and for $a = 1$ via Airy function. At large times $F(\zeta_1,\zeta_2) = \exp[\lambda_0(\zeta_1 + \zeta_2)/2]\Psi_0(\zeta_1 - \zeta_2)$. If ζ_{char} is the width of Ψ_0 then by Heisenberg uncertainty principle, $\zeta_{char}^{2}\beta_1\zeta_{char}^{a} \sim 1$ and we obtain $\zeta_{char} \sim \beta_1^{-1/(2+a)}$. In particular, for smooth potential, $a = 2$ and $\zeta_{char} \propto \beta_1^{-1/4}$ while for linear potential, $a = 1$ and $\zeta_{char} \propto \beta_1^{-1/3}$. Note that if $\beta_1 \sim \beta_2$ then for any a one has $\zeta_{char} \gg \lambda_0^{-1}$ since $\lambda_0 \sim \beta_2^{-1/2}$. We see that the width of $\Psi_0(\zeta_1 - \zeta_2)$ (determined by the stretching rate in a fast case) in a slow case depends on the different-time strain correlation function at time differences small in comparison with τ. The quantity $\tau_s \equiv \tau\zeta_{char}$ is the stretching correlation time which due to $\zeta_{char} \ll 1$ is much smaller than τ in the slow case for any finite a. Only for $a \gg 1$ when $g(t/\tau)$ is practically constant until $t \simeq \tau$ we have $\tau_s \sim \tau$. Using the uncertainty principle and the normalization condition we obtain $s_0 \sim \beta_1^{2/(2+a)}$ and $\Psi_0^2 \sim \beta_1^{1/(2+a)}$. Thus, $s_0 \ll \beta_1$ and, consequently, $\lambda_0 = \sqrt{\beta_2 - s_0} \simeq \sqrt{\beta_2}$ if $\beta_1 \sim \beta_2$.

Let us now estimate the cumulants of R^2. To find $\bar F(\zeta_1,\zeta_2,\zeta)$ we rewrite Eq. (26) in terms of $\zeta_+ = (\zeta_1+\zeta_2)/2$ and $\zeta_- = \zeta_1 - \zeta_2$: $[\partial_+^2/4 - \partial_-^2 - \beta_2 + \beta_1|\zeta_-|^a]\bar F_{12} = -2\delta(\zeta_-)\delta(\zeta-\zeta_+)$. $\bar F_{12} = 0$ if $\zeta_1 > \zeta$ or $\zeta_2 > \zeta$. At a given $\zeta - \zeta_+$, $\bar F_{12} = 0$ at large enough $|\zeta_-|$, so it can be expanded into the series $\bar F_{12} = \sum_n hn(\zeta-\zeta_+)\Psi_n(\zeta_-)$, where $(\partial_\zeta^2/4 - \beta + s_n)h_n(\zeta - \zeta_+) = -2\Psi_n^*(0)\delta(\zeta-\zeta_+)$ and $h_n = 0$ at $\zeta < \zeta_+$.

Solving the last equation, we get $\bar{F}_{12} = -4\theta(\zeta-\zeta_+)\sum_n^* \Psi n(0)\Psi_n(\zeta_-)[e^{\lambda_n(\zeta-\zeta_+)} - e^{-\lambda_n(\zeta-\zeta_+)}]/\lambda_n$ with $\lambda_n = 2\sqrt{\beta_2 - s_n}$. Asymptotically at $\zeta-\zeta_+ \gg \zeta_{\rm char}$ only the first term $-4\lambda_0^{-1}\Psi_0(0)\Psi_0(\zeta_-)\exp[\lambda_0(\zeta-\zeta_+)]$ contributes to $\bar{F}_{12}$. Thus, at $|\zeta_-| < \zeta_{\rm char}$ we obtain the estimate $\bar{F}_{12} \sim \beta_1^{1/(2-a)}\beta_2^{-1/2}\exp[\lambda_0(\zeta-\zeta_+)]$. For arbitrary β, one can find the useful integral representation for $F(\zeta_5,\zeta_5,\zeta_4) = F_{554}$. We multiply Eq. (25) by $\bar{F}(\zeta,\zeta_1,\zeta_5)$, integrate over ζ,ζ_1 and use Eq. (26): $2F_{554} = \beta^2\int d\zeta_0\, d\zeta_1\, d\zeta_2\, d\zeta_3 \bar{F}_{015}\bar{F}_{234}F_{02}g(\zeta_0-\zeta_2)F_{13}g(\zeta_1-\zeta_3)$. At $\zeta_5,\zeta_4 \gg 1$, the main contribution to the integral is determined by the region $1 \ll \zeta_0 \approx \zeta_1 \approx \zeta_2 \approx \zeta_3 \ll \min(\zeta_4,\zeta_5)$, since, e.g. $\bar{F}_{234} = 0$ if $\zeta_2 > \zeta_4$ or $\zeta_3 > \zeta_4$. The integration over three differences gives $\sim \zeta_{\rm char}^3$ whereas the fourth integration gives $\min(\zeta_4,\zeta_5)$. For the slow case, the ratio $F(\zeta_5,\zeta_5,\zeta_4)/dF(\zeta_4,\zeta_4)F(\zeta_5,\zeta_5)$ thus gives

$$\frac{\langle R^2(\zeta_4)R^2(\zeta_5)\rangle - \langle R^2(\zeta_4)\rangle\langle R^2(\zeta_5)\rangle}{\langle R^2(\zeta_4)\rangle\langle R^2(\zeta_5)\rangle} \sim \frac{\beta^{(1+a)/(2+a)}}{d}\min(\zeta_4,\zeta_5)\,, \tag{30}$$

where we assumed $\beta_1 \sim \beta_2 \sim \beta$. As well as in the fast case, we encounter the secular growth of the cumulant. Large-d approach is valid when RHS of Eq. (30) is small, i.e. at $\zeta < d/\beta^{(1+a)/(2+a)}$. If $d/\beta^{(1+a)/(2+a)} \gg 1$ then there exists the interval $1 \ll \zeta \ll d/\beta^{(1+a)/(2+a)}$ when both Eqs. (30) and Eq. (8) work. Comparing Eq. (9) with Eq. (30) we conclude $\bar{\lambda} \sim \sqrt{\beta}/\tau \sim \sqrt{D/\tau}$ and $\Delta \sim \beta^{(1+a)/(2+a)}/d\tau \sim \bar{\lambda}^2\tau_s/d$, contrary to a naive expectation $\Delta \sim \bar{\lambda}^2\tau$ suggested in Ref. [9].

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