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Nonstationary Wave Turbulence

G. E. Falkovich[1] and A. V. Shafarenko[2]

[1] Department of Nuclear Physics, Weizmann Institute of Science, Rehovot 76100, Israel. Permanent address: Institute of Automation and Electrometry, Novosibirsk 630090, Russia, USSR

[2] Department of Electronic and Electrical Engineering University of Surrey, Guildford GU2 5XH, UK

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Summary. Nonstationary regimes of the wave turbulence evolution are considered in the framework of isotropic kinetic equation. It is predicted analytically and confirmed by numerical experiment that there is a class of wave systems in which any initial distribution of the turbulence energy in k-space comes into a universal, Kolmogorov-type spectrum in a finite time. Before and after the formation of the Kolmogorov spectrum, two different self-similar regimes of evolution occur: the first one is responsible for explosively forming the universal spectrum and the second one determines energy dissipation.

Key words. wave turbulence, Kolomogorov spectrum, self-similarity, explosion regime

1. Introduction

Few would not argue that the problem of turbulence stretches beyond the framework of hydrodynamics of ideal incompressible fluid. It pertains to many other physical topics ranging from collective oscillations in solids and plasmas to such global processes as galaxy formation. At present, one can describe turbulence as a strongly excited state of any dissipative system with a large number of degrees of freedom that has to be described statistically and which is a long way from thermodynamic equilibrium.

Turbulence appears in a few forms in these media, and it manifests itself in a few ways in hydrodynamics. Basically, the variety of occurrences of turbulence in nature can be divided into two classes:

1. Vortex turbulence, and
2. Wave turbulence.

The first class includes, but is not restricted to, the classical turbulence of incompressible fluid, where no small-amplitude waves are found against a homogeneous background. There can be no consistent linearization of the dynamic equations for this type of turbulence, which are therefore strongly nonlinear however small the level of excitation.

The physical systems that do allow a consistent linear approximation describing small-amplitude waves with a dispersion belong to the second class. Feasibility of such an approximation implies that one can imagine a situation in which linear effects expressly predominate over nonlinear ones: dispersion of phase velocity has a stronger effect than interaction between waves [1]. This sort of turbulence is called weak. It is amenable to an efficient theoretical analysis in terms of closed kinetic equations (see Eqs. 5 and 8 below). An advanced theory of weak turbulence considers various wave phenomena in plasma, solids, atmosphere, ocean, and space [1].

Although the two above classes of turbulent phenomena are different, still one can draw a number of conceptual parallels between the respective theories. Notably, the theory of hydrodynamic turbulence incorporates into its foundations the "cascade hypothesis" by Richardson-Kolmogorov-Obukhov [2] (and also, later and independently, by Onsager [3], Heizenberg [4] and von Weizsacker [5]) which prompts a scale-invariant structure of the energy distribution among vortices. V. E. Zakharov made the same presumption later in the theory of wave turbulence. His intention was to find steady-state spectra as exact solutions of kinetic equations, which he did using special transforms [1], which later found application to vortex hydrodynamics [6].

By this time the theory of stationary spectra of wave turbulence, which includes both stability theory [1,9,10] and the problem of matching the Kolmogorov solutions with sources and sinks [7,8], is well developed.

In this paper we present the theory of nonstationary wave turbulence. We will analyze both the free evolution regimes that appear when no external source is in effect and the processes of forming Kolmogorov spectra after switching the source on.

We shall discuss two physical concepts playing an important role in the analysis of nonstationary turbulence. One of these is the concept of stationary cascade spectra of the Kolmogorov type. In nonstationary regimes, which are the subject of this paper, the Kolmogorov spectrum is realized either as an asymptotically stable state (i.e., one that an arbitrary distribution tends to in the presence of a source and a sink) or as an intermediate asymptotics (for freely decaying turbulence). The other concept is self-similar regimes of evolution, which can be of two kinds:

1. Those that form for the long time (formally at $t \to \infty$), and
2. Quickly forming and living for a finite time.

The latter case corresponds to explosion regimes of evolution, which proceed in self-accelerating manner. As we shall show, bi-self-similar regimes, where one of the asymptotics of the distribution forms in a self-accelerating (explosion) way for a finite time, and then the evolution of the distribution as a whole goes on by a decelerating self-similar law, are also possible. It should be noted that the concept of bi-self-similar regimes has been used recently for describing wave collapses and self-focusing [11] as well as other nonlinear phenomena.

2. Hamiltonian Formalism and the Kinetic Equation

The various occurrences of turbulence in conservative or slightly dissipative systems can be described from a single viewpoint with the help of Hamiltonian formalism [12]. To use it is convenient because the waves of quite different nature are described by a single equation:

$$\frac{\partial a(\mathbf{k}, t)}{\partial t} = -i \frac{\delta \mathcal{H}}{\delta a^*(\mathbf{k}, t)} \tag{1}$$

where $a(\mathbf{k}, t)$ is the amplitude of the wave with some wavevector $\mathbf{k}$, and $\mathcal{H} = \mathcal{H}\{a, a^*\}$ is the Hamiltonian, which is a functional of the wave amplitudes. For weakly nonlinear wave systems, the Hamiltonian can be expanded into a series in the wave amplitude, where a few nonvanishing terms can be held. As long as the medium is assumed to be in equilibrium at zero wave amplitudes, the expansion begins with a quadratic term, which may be written as follows:

$$\mathcal{H}_2 = \int \omega(\mathbf{k}) a(\mathbf{k}, t) a^*(\mathbf{k})\, d\mathbf{k}. \tag{2}$$

(We refer the interested reader to [1] and [12] for details.) This term generates an equation of motion

$$\frac{\partial a(\mathbf{k}, t)}{\partial t} = -i\omega(\mathbf{k}) a(\mathbf{k}),$$

from which it is clear that $\omega(\mathbf{k})$ is the wave frequency. Thus, all we need to be able to describe various linear effects such as interference, diffraction, and so on is to know the single function $\omega(\mathbf{k})$, usually called the wave dispersion law.

Similarly, all the information about interaction is contained in the consequent terms of the Hamiltonian expansion. The cubic term

$$\mathcal{H}_3 = \int \big[V(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) a^*(\mathbf{k}) a(\mathbf{k}_1) a(\mathbf{k}_2) + V^*(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) a(\mathbf{k}) a^*(\mathbf{k}_1) a^*(\mathbf{k}_2)\big] \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)\, d\mathbf{k}\, d\mathbf{k}_1\, d\mathbf{k}_2 \tag{3}$$

describes three-wave processes of decay and confluence. The respective equation in $a(\mathbf{k})$ becomes nonlinear in that case

$$\frac{\partial a(\mathbf{k})}{\partial t} + i\omega(\mathbf{k}) a(\mathbf{k}) = -i \int \big(V_{\mathbf{k}12} a_1 a_2 + 2V^*_{1\mathbf{k}2} a_1 a_2^*\big) \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)\, d\mathbf{k}_1\, d\mathbf{k}_2. \tag{4}$$

Here and below, a shorthand notation of arguments as indices is used. To introduce a statistical description, which is adequate only to turbulence fields, one has to fulfill the following program [1].

1. Let us introduce the interaction representation for the dynamic variable: $a_{\mathbf{k}}(t) = c_{\mathbf{k}}(t)\exp(-i\omega_{\mathbf{k}}t)$ in order to extract a "slow" evolution of the field caused by interaction between waves.
2. Because the phases of the waves do not contribute to the Hamiltonian $\mathcal{H}_2$ and the subsequent terms of the Hamiltonian expansion are assumed to be small, it is possible to consider the phases as being randomized by themselves or by fluctuations of the medium, with such randomization having a negligible effect on the total energy. Thus we arrive at the random phase approximation [1] that assumes that the statistics of the field is Gaussian. Taking into account again the fact that the medium in question is spatially homogeneous we introduce a statistical characteristic $n(\mathbf{k}, t)$ averaged over the ensemble of random phases such that

$$n(\mathbf{k}, t)\delta(\mathbf{k} - \mathbf{k}') = \langle c(\mathbf{k}, t)c^*(\mathbf{k}', t)\rangle.$$

The pair correlator $n(\mathbf{k}, t)$ evolves according to the following kinetic equation:

$$\begin{aligned}\frac{\partial n(\mathbf{k}, t)}{\partial t} = &\int \left|V_{\mathbf{k}12}\right|^2 \delta(\omega_{\mathbf{k}} - \omega_1 - \omega_2)\,\delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)\\ &\times [n_1 n_2 - n_{\mathbf{k}}(n_1 + n_2)]\, d\mathbf{k}_1\, d\mathbf{k}_2\\ &- 2\int \left|V_{1\mathbf{k}2}\right|^2 \delta(\omega_1 - \omega_{\mathbf{k}} - \omega_2)\,\delta(\mathbf{k}_1 - \mathbf{k} - \mathbf{k}_2)\\ &\times [n_{\mathbf{k}} n_2 - n_1(n_{\mathbf{k}} + n_2)]\, d\mathbf{k}_1\, d\mathbf{k}_2. \end{aligned} \tag{5}$$

To derive this equation one should rewrite (4) in the interaction representation, multiply it by $c^*_{\mathbf{k}}$, add the complex conjugated equation, and average the result. The triple correlator $\langle c^*_{\mathbf{k}} c_1 c_2\rangle$ can be found from a similar equation derived using the procedure just mentioned except that the multiplication has to be by a factor $c^*_1 c_2$. Eventually, the triple correlator happens to be coupled with the fourth-order one, with the latter split into the product of pair correlators:

$$\langle c^*_1 c^*_2 c_3 c_4\rangle = n_1 n_2 [\delta(\mathbf{k}_1 - \mathbf{k}_3)\,\delta(\mathbf{k}_2 - \mathbf{k}_4) + \delta(\mathbf{k}_2 - \mathbf{k}_3)\,\delta(\mathbf{k}_1 - \mathbf{k}_4)]\,.$$

The kinetic equation (5) is valid (and the statistical properties used in its derivation actually take place) in the limit of small nonlinearity, where the characteristic time of interaction $t_a \propto \omega(|V|^2 k^d n)^{-1}$ is much greater than the time of the wave packet diffusion determined by dispersion $t_D \propto (k^2\omega''_k)^{-1}$.

Equation 5 makes sense if the conditions imposed by δ-functions are compatible, that is, the dispersion law enables spatio-temporal resonance:

$$\omega(\mathbf{k}_1) + \omega(\mathbf{k}_2) = \omega(\mathbf{k}_1 + \mathbf{k}_2)\,. \tag{6}$$

Such a dispersion law is termed a decay dispersion law. If the three-wave processes are nonresonant, the right-hand side of (5) is identically equal to zero. For the waves with such (nondecay) dispersion laws one should expand the Hamiltonian up to the fourth-order terms. The Hamiltonian that appears after elimination of nonresonant terms [1] will have the following simple view:

$$\mathcal{H}_4 = \int T_{1234} a_1^* a_2^* a_3 a_4 \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4)\, d\mathbf{k}_1\, d\mathbf{k}_2\, d\mathbf{k}_3\, d\mathbf{k}_4 \tag{7}$$

and will describe the wave-scattering processes. The respective kinetic equation reads:

$$\frac{\partial n(\mathbf{k},t)}{\partial t} = \int |T_{1234}|^2 \delta(\omega_{\mathbf{k}} + \omega_1 - \omega_2 - \omega_3)\delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) \times (n_1 n_2 n_3 + n_{\mathbf{k}} n_2 n_3 - n_{\mathbf{k}} n_{\mathbf{k}} n_1 - n_{\mathbf{k}} n_1 n_3)\, d\mathbf{k}_1\, d\mathbf{k}_2\, d\mathbf{k}_3. \tag{8}$$

It is not difficult to prove that in an isotropic medium (i.e., such that $\omega(\mathbf{k}) = \omega(k)$) for the Goldstone mode (i.e., such that $\omega(0) = 0$), the dispersion law is decay if and only if

$$\omega'' = \frac{d^2\omega(k=0)}{dk^2} > 0$$

(see, e.g., [6] for details). We shall consider only isotropic media and isotropic distributions of the statistical fields.

The number of stationary solutions (precisely, the number of independent parameters in the general solution) is equal to the number of independent integrals of motion. The three-wave kinetic equation (5) generally has only one integral of motion: the energy $E = \int \omega_k n_k\, dk$. There is one and only solution that corresponds to a constant energy flux in k-space. It is called the *Kolmogorov spectrum* and can be derived analytically for the case of scale-invariant wave systems where $\omega(k) = ck^\alpha$ and $V(k, k_1, |\mathbf{k} - \mathbf{k}_1|) = k^m V_0 f(k_1/k)$ (here f is a dimensionless structural function). In that case $n_k \propto k^{-s_0}$, $s_0 = m + d$, and d is the dimensionality of the k-space [1]. If the Kolmogorov distribution falls off faster than the equilibrium one $n_k \propto \omega_k^{-1}$, that is, if $m + d > \alpha$, then the energy flux is directed toward large k. For the case of four-wave interaction, the total number of waves (or "the wave action") in the system $N = \int n_k\, d\mathbf{k}$ is conserved as well as the energy. Assuming scale-invariance: $T(\lambda k, \lambda k_1, \lambda k_2, \lambda k_3) = \lambda^m T(k, k_1, k_2, k_3)$, we can derive two stationary solutions:

1. $n_1(k) \propto k^{-s_1}$, where $s_1 = d + 2m/3$, with the energy flux, and
2. $n_2(k) \propto k^{-s_2}$, where $s_1 = d + 2m/3 - \alpha/3$, with the flux of the wave action.

Usually $s_1 > s_2 > 1$, so the energy flux is directed to the short-wave region, while the wave action flux to the long-wave one [13].

The validity of kinetic equation should also be checked for Kolmogorov solutions. For example, substituting $n(k) \propto k^{-s_0}$ into the estimate of the characteristic interaction time gives $t_a \propto k^{s_0+\alpha-d-2m}$. Comparing it with the dispersion time $t_D \propto k^{-\alpha}$, one can see that the validity of weak turbulence approximation improves while energy cascades toward small scales if $s_0 + 2\alpha > d + 2m$. If the opposite inequality takes place, turbulence becomes stronger in the small-scale region. In the same way, one could check the weakness of interaction for Kolmogorov spectra in the case of four-wave kinetic equation. Also, it should be noted that we consider here a *slow* evolution yet for *finite* time. For sufficiently long time the high-order nonlinear processes which have been neglected may distort the picture of weak turbulence. For example, the conservation of wave action can be violated.

3. Underlying Physical Models

All the subsequent analysis will be as general as possible, the results to be applied to the following physical systems [1]:

A. *The decay case.*
   1. Capillary waves on deep ($d = 2$, $\alpha = \frac{3}{2}$, $m = \frac{9}{4}$) and shallow ($d = 2$, $\alpha = 2$, $m = 2$) water.
   2. The three-dimensional sound with a positive dispersion $\omega(k) = ck(1 + a^2k^2)$, $ak \ll 1$ ($d = 3$, $\alpha = 1$, $m = \frac{3}{2}$)
   3. The two-dimensional sound with a positive dispersion, for example, gravitational-capillary waves on shallow water ($d = 2$, $\alpha = 1$, $m = 1$).

B. *The nondecay case*
   1. Gravitational waves on deep water ($d = 2$, $\alpha = \frac{1}{2}$, $m = 3$).
   2. Langmuir waves in plasmas and spin waves in magnetic materials ($d = 3$, $\alpha = 2$, $m = 0$ or $m = 2$ depending on the type of interaction).
   3. The three-dimensional sound with a negative dispersion, for example, ion sound in plasma $\omega(k) = ck(1 - a^2k^2)$, $ak \ll 1$ ($d = 3$, $\alpha = 1$, $m = 1$).
   4. The two-dimensional sound with a negative dispersion, for example, gravitational waves on shallow water $\omega(k) = ck(1 - a^2k^2)$, $ak \ll 1$ ($d = 2$, $\alpha = 1$, $m = 2$).

4. Empirical Approach

We shall start considering the evolution of wave distributions with simple dimensional estimations, which help understanding of the dependence of the characteristic evolution time on the mean wavelength. Let us consider the cascade transfer of waves from large scales to smaller ones. As we shall see now, the character of evolution dramatically depends on the sign of the exponent $h = \alpha + d - s_0$. To demonstrate that is true, let us first consider the evolution of small perturbations against the background of the stationary spectrum $n_k \propto k^{-s_0}$ carrying an energy flux. Such perturbations evolve according to the kinetic equation linearized with respect to $\delta n(k, t)$:

$$\frac{\partial \delta n(k,t)}{\partial t} = \hat{L}(k, k_1)\delta n(k_1, t). \tag{9}$$

Here operator $\hat{L}(k, k_1)$ is expressed in terms of the frequency, interaction coefficient, and spectrum $n(k)$, thus being scale invariant (see [1] for details). The scaling exponent of this operator could be established by the following simple derivation. The exponent of the right-hand side of the complete kinetic equation with $n(k) \propto k^{-s_0}$ is equal to $-\alpha - d$, since after multiplying by ω_k and integrating over $d\mathbf{k}$ it should ensure that the energy flux does not depend on k. Since the exponent of the complete equation differs from that of the linearized operator by s_0, then the latter is equal to $s_0 - \alpha - d = -h$. So if the distribution deviates from the stationary one at some k, the typical time of variations will be

$$t_{\mathrm{NL}}^{-1} \propto k^{-h} \tag{10}$$

When external influence is absent, the energy must be conserved, therefore the characteristic time may be evaluated from the kinetic equation as follows:

$$t_{\mathrm{NL}}^{-1} \propto k^{-2h} E, \tag{11}$$

where $E = \int \omega_k n_k \, d\mathbf{k}$ is the total energy of the distribution. In particular, in both cases the process of wave transfer to large k accelerates or decelerates depending on the sign of h. It is appropriate to remember here one more role of the h value: it points out the edge of the Kolmogorov distribution at which the major part of the energy of the turbulence will concentrate:

$$E = \int \omega_k n_k^0 \, d\mathbf{k} \propto k^h. \tag{12}$$

For example, at $h > 0$ major part of the Kolmogorov spectrum energy is confined in the region of large k. In this case, as we can see from (10) and (11), the evolution slows down, with the distribution drifting towards greater wave numbers. Vice versa, the motion of the distribution to the region opposite to the energy-containing one is an accelerated process.

In Sect. 5 we shall first discuss nonstationary behavior of weak turbulence distributions of waves with a decay dispersion law, when $s_0 = m + d$ and $h = \alpha - m$. Since h is equal to the difference between the exponents of the frequency and the interaction coefficient, its sign indicates which coefficient of the Hamiltonian grows more quickly with k, whether the one responsible for the linear phenomena or for the interaction. As will be evident from the dimensional analysis we shall give in Sect. 5, the motion of a distribution initially localized in the long-wave region towards large k proceeds in a self-similar manner. After switching a long-wave source on, the stationary Kolmogorov distribution starts forming with a self-similar relaxation front, too. The difference in the behavior of two wave systems with opposite signs of h is that at $h < 0$ the front of spectrum formation moves explosively, that is, reaches infinity for a finite period of time, whereas at $h > 0$ it does not.

A strict analytical proof of the explosive character of pumping over is given in Sect. 6 for the particular case of a weak three-dimensional acoustic turbulence ($h = -\frac{1}{2}$). The idea of the proof is to consider the temporal variation of the moments of the distribution function in the k-space. Proceeding from the kinetic equation, one can prove that if some moments are finite initially (i.e., the distribution is initially rapidly decreasing), then for a finite time those moments will become infinite, which corresponds to a power asymptotics forming at $k \to \infty$. For the two-dimensional acoustic turbulence, which presents an intermediate case $h = 0$, that section gives analytical proof of a nonexplosive character of evolution. Section 7 gives the results of our numerical simulation that convincingly supports the ideas disclosed in Sects. 5–6.

5. Analysis of Self-similar Substitutes

We start with the three-wave kinetic equation (5). For the sake of simplicity we assume the distribution to be isotropic. In this case, there is only one integral of motion, *viz.* the energy. We shall assume that $m + d > \alpha$, that is, after establishing itself, the Kolmogorov distribution transfers energy to the short-wave region. Two physically different statements of the problem will be discussed:

P1: Expansion of the Kolmogorov distribution into the region of large k, and

P2: Decaying turbulence: free evolution of the initially long-wave packet which, owing to a tendency to equilibrium distribution, should also spread all over the k-space.

It would be natural to presume that the evolution will in some time become self-similar a long way from the source or the initial localization site of a wave packet. Let us discuss possible self-similar substitutions in the three-wave kinetic equation (5), that is, such that have the following form:

$$n(k, t) = t^{-q} f(kt^{-p}) = t^{-q} f(\xi). \tag{13}$$

We shall consider here the ξ variable to be dimensionless, assuming that k is measured in k_0 units (where k_0 is the location of the source or the initial location of the packet), and t in the units of

$$t_{\mathrm{N}} = \frac{V_0^2 k_0^{2m+d} n^2(k_0)}{\omega(k_0)},$$

where t_{N} is the characteristic time of the nonlinear interaction in the region $k \simeq k_0$ and V_0 is the dimensional constant associated with the interaction coefficient.

Substituting (13) in the three-wave kinetic equation

$$\frac{\partial n(k, t)}{\partial t} = I(k), \tag{14}$$

we obtain

$$-(gf + p\xi f') = I(\xi) t^{p(2m-\alpha+d)-q+1}.$$

It is evident from this that the solution of the form (13) can exist only if a condition

$$p(2m - \alpha + d) - q + 1 = 0$$

is satisfied.

To obtain another relationship between the parameters p and q, we should specify which of the above cases *P1* and *P2* we are going to address.

Case P1. The quasistationary Kolmogorov distribution $n_k^0 \propto k^{-m-d} = k^{-s_0}$ must occur between the source and the relaxation front. This means that $f(\xi) \propto \xi^{-m-d}$ at $\xi \to 0$ and the condition of steadiness gives

$$p(m + d) = q. \tag{15}$$

From (5) and (15), we find $q = (m + d)/(\alpha - m) = s_0/h$ and

$$p = (\alpha - m)^{-1} = 1/h, \tag{16}$$

which is consistent with estimate (10). The boundary of the Kolmogorov distribution corresponds to $\xi \simeq 1$ and moves in the k-space by the law $k_b \propto t^p = t^{1/h}$. So the solution (13) in this case describes the evolution of the Kolmogorov spectrum toward large k only if $h > 0$. The same conclusion can be drawn by considering the kinetic equation in the self-similar variables

$$-(qf + p\xi f') = I(\xi), \tag{17}$$

Setting in (17) $f(\xi) \propto \xi^{-m-d}$, we can see that $I(\xi)/f(\xi) \propto \xi^{-h}$, that is, the $I(\xi)$ term prevails in the region $\xi \ll 1$ (and that $f(\xi)$ has the Kolmogorov asymptotics there) at $h > 0$. The logarithmic front velocity,

$$\frac{d \ln[k_b(t)/k_b(0)]}{dt} \propto (ht)^{-1},$$

decreases with time, in agreement with the general idea that the process of the distribution expansion is a slowing-down process when expanding the energy-containing region.

At $h \to 0$, the front velocity dramatically increases and becomes infinite at $h = 0$. This implies that at $h < 0$ the rate of formation of the Kolmogorov spectrum increases with time so quickly that the relaxation front reaches infinity for a finite time. Indeed, at $h < 0$, in order to obtain a self-similar relaxation front moving to large k, one should replace t by $\tau = t_0 - t$ in the right-hand side of (13):

$$n(k, t) = \tau^{-q} f(k\tau^{-p}). \tag{18}$$

The equations (5–16) for p and q will retain their form, but now the right boundary of the Kolmogorov distribution corresponding to $k_b\tau^{-p} \simeq 1$ moves by the explosion law $k_b \propto \tau^{1/h}$ and reaches infinity for the finite time t_0 determined by the initial distribution (we shall explain how a little later). Qualitative understanding of t_0 being finite lies with the fact that wherever $h < 0$ the stationary spectrum energy is concentrated in the long-wave region, which means that an interval $[k, \infty]$, $k \neq 0$, contains finite energy. This finite energy obviously takes finite time to become redistributed over the interval.

The reader may wonder if the weak turbulence approximation allows one to describe an explosive regime of the spectrum evolution toward the Kolmogorov cascade. Indeed, such an approximation implies that the typical time of pair correlator variations is much larger than the dispersion time (which is of the order of the inverse frequency), whereas the explosive process goes in a self-accelerating way with the characteristic interaction time decreasing with k. However, the dispersion time decreases with k as well, due to the increase in frequency ω_k. So the applicability condition for an explosive regime of spectrum forming depends on which of the times decreases faster. That condition obviously coincides with the one on Kolmogorov spectrum itself (see also the ends of Sect. 2 and Sect. 6).

It is interesting to examine the behavior of the energy accumulated in the self-similar part of the distribution. For the solutions (13) and (18) we have, respectively,

$$E(t) = t^{2ph-1} \int_0^\infty f(\xi)\xi^{\alpha+d-1}\, d\xi \tag{19}$$

and

$$E(t) = \tau^{2ph-1} \int_0^\infty f(\xi)\xi^{\alpha+d-1}\, d\xi. \tag{20}$$

The fact that $p = 1/h$ leads to $E \propto t$ at $h > 0$ and $E \propto \tau = t_0 - t$ at $h < 0$. The linear growth of energy in the system with $h > 0$ means that the self-similar solution forms when the occupation numbers of the waves influenced directly by the source do not change any longer and a constant energy flux into the system has been established. In that case, the main part of energy is concentrated in the self-similar region. On the contrary, at $h < 0$ the proportion of energy contained in the solution (18) decreases as the self-similar wave moves to the short-wave region. Thus, the universal set-in regimes of the Kolmogorov spectra in the cases $h > 0$ and $h < 0$ are realized with absolutely different time scales. At $h > 0$ the self-similar front (13) is formed for the time much larger than the typical stabilization time for the occupation numbers in the pump region. In the case $h < 0$, the self-similar front appears for a small period of time ($t_0 - t \ll t_0$), being too small for any essential changes in the occupation numbers of long waves to occur.

Case P2. We shall start analysis of the free evolution with the case $h < 0$. Since the front (18) is self-accelerating

$$\frac{1}{k_b}\frac{dk_b}{dt} \propto \frac{1}{\tau},$$

the behavior of the short-wave part of the distribution should be insensitive to the presence or absence of a long-wave source that changes the occupation numbers not faster than exponentially. So until $t = t_0$ the free expansion of the distribution to large k proceeds in the same way as it does with the source: the self-similar front $n(k, t) = \tau^{-q} f(k\tau^{-p}) = \tau^{-q} f(\xi)$ with $q = s_0/h$ and $p = 1/h$ moves explosively, leaving behind the Kolmogorov distribution, $f(\xi) \propto \xi^{s_0}$ at $\xi \ll 1$. Ahead of the front, the occupation numbers should quickly diminish with ξ (and, accordingly, k). It was assumed in [14] that if the waves in the short-wave region of the distribution interact mainly with each other, then the asymptotics is exponential: $n_k \propto \exp[-(\omega_k/\omega_b)] = \exp[-(k/k_b)^\alpha]$. Let us find out the conditions for this to be possible. Introducing the variable $\eta = \xi^\alpha$, we write the kinetic equation (17) in the form

$$-[qf(\eta)+\alpha p\eta f'(\eta)] = \int_0^\eta [\eta_1(\eta-\eta_1)]^{d/\alpha-1}\eta^{2m/\alpha} f_1^2(\eta_1/\eta)$$
$$\times \Delta_d^{-1}[f(\eta_1)f(\eta-\eta_1)f(\eta)f(\eta_1)-f(\eta)f(\eta-\eta_1)]d\eta_1$$
$$-2\int_\eta^\infty [\eta_1(\eta_1-\eta)]^{d/\alpha-1}\eta_1^{2m/\alpha} f_1^2(\eta/\eta_1)\Delta_d^{-1}[f(\eta)f(\eta_1-\eta)$$
$$-f(\eta_1)f(\eta_1-\eta)-f(\eta_1)f(\eta)]\,d\eta_1. \tag{21}$$

Here f_1 is a structural function of the interaction coefficient, which, being expressed via the frequency ratio $x=\omega_1/\omega$, has the following properties: $f_1(x)=f_1(1-x)$ and $f_1(x)\propto x^{m_1/\alpha}$ at $x\to 0$. The Δ_d^{-1} value appears as a result of angle averaging of the δ-function of wave vectors [6]. Let us consider (21) at $\eta\gg 1$ and set $q=s_0/h$, $p=1/h$, $f(\eta)=\eta^{-b}\exp(-\eta)$. Using the asymptotics of the function $f_1(x)$ at $x\to 0$, we obtain from (21):

$$\alpha\eta^{1-b}e^{-\eta} = -2he^{-\eta}\eta^{(1-\alpha+2m-2m_1)/\alpha-b}\int_1^\eta \eta_1^{(d-1-\alpha+2m_1)/\alpha-b}(1-e^{-\eta_1})^2\,d\eta_1.$$

Hence, such asymptotics can exist only if the inequality

$$m_1-m+\alpha\geq \tfrac{1}{2}$$

is satisfied. For example, for the capillary waves on a deep fluid, that condition is satisfied ($h=\frac{3}{4}$, $\alpha=\frac{3}{2}$, $m=\frac{9}{4}$, $m_1=\frac{7}{2}$), while for the three-dimensional sound not ($h=-\frac{1}{2}$, $\alpha=1$, $m=\frac{3}{2}$, $m_1=\frac{1}{2}$). The question of finding the self-similar asymptotics analytically in the region $\xi\geq 1$ with the condition (5) being violated is open. The numerical simulation carried out for the three-dimensional sound (see Fig. 1) shows that again n_k falls off with k ahead of the front steeper than it would do conforming to the Kolmogorov law.

Since for $t<t_0$ the asymptotics of the distribution at $k\to\infty$ falls off steeper, the total energy is conserved. What happens at $t\geq t_0$, that is, after the right boundary of the Kolmogorov distribution has reached infinity? From that moment on, $n_k\propto k^{-s_0}$ at $k\to\infty$, the energy flux at $k=\infty$ being nonzero, therefore the energy has to diminish. Any real system, of course, has a sink at a finite k_m and the energy will start to diminish somewhat earlier than at the moment t_0, namely, when the relaxation front reaches the sink. It would be natural to presume that after a large enough period of time ($t\gg t_0$), the process of attenuation will come into a second self-similar regime with the energy diminishing by a power law. Such a behavior should be described by the self-similar solution of the form (13). The relation between q and p is given by (5), and the energy decreases by the law (19). To determine the exponent p, one has to solve a nonlinear eigenvalue problem for (17) with an additional requirement $f(\xi)\geq 0$ at $0<\xi<\infty$. Proceeding from the energy decrease condition, one can impose the following limitation on p:

$$p>(2h)^{-1}.$$

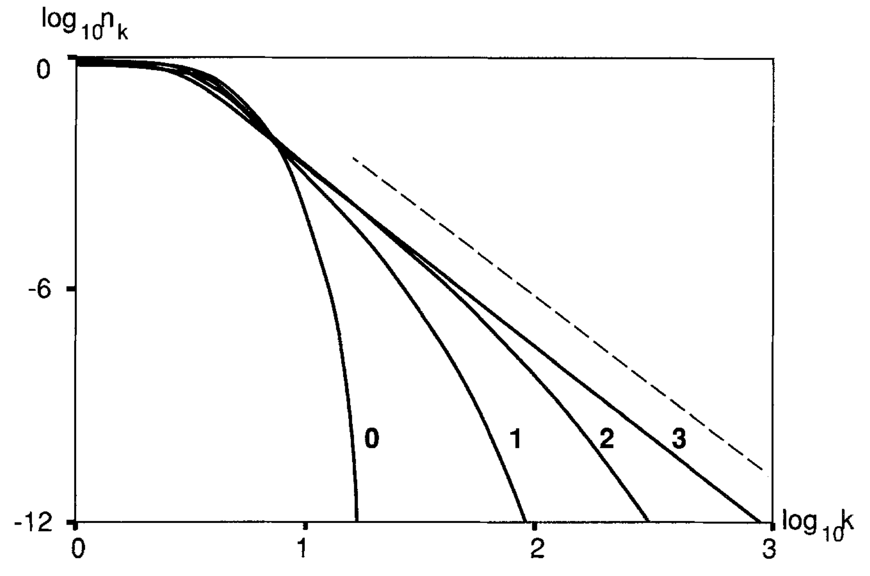

Fig. 1. Distribution of occupation numbers n_k in k-space at different moments in time for the case of 3d sound turbulence. Curves 0–3 represent $\ln n(k)$ at the moments in time 0, 10^{-4}, $3 \cdot 10^{-4}$ and $5.3 \cdot 10^{-4}$, respectively. The dashed line represents the Kolmogorov power law $n(k)^0 = Ak^{-9/2}$.

Since we consider the case of negative h, this inequality allows for both signs of p, $p > 0$ and $p < 0$; solutions with a positive p describe a distribution moving to the right, and those with a negative p describe distribution moving to the left. However, since the sink (be it $k = k_m$ or $k = \infty$) is stationary, we can *presume* that, the distribution as a whole should dissipate without drifting anywhere, that is, $p = 0$ and $E \propto t^{-1}$. The law of energy decrease may also be obtained from a chain of estimates based on the same presumption $dE/dt \propto P \propto E^2 \Rightarrow E \propto t^{-1}$. The results of numerical simulations (see Fig. 2) are consistent with that presumption. Thus, at $h < 0$ the behaviour is bi-self-similar.

The free evolution of a system with $h > 0$ in the large-k region also should at some time come into a self-similar solution of the type (13). But now the energy must be conserved, therefore the exponent p is determined unambiguously: $p = (2h)^{-1}$ which is consistent with estimate (11). The positiveness of p implies that the distribution drifts to the short-wave region. Such a solution has no Kolmogorov asymptotics anywhere since the energy flux is equal to zero at $k \to 0$ and $k \to \infty$.

Now we would like to emphasize the profound difference between the systems with $h > 0$ and $h < 0$. In the former case, the energy of freely decaying turbulence is always conserved provided there is no dissipation. On the contrary, for the case $h < 0$ the finite rate of energy dissipation arises after a finite time even without any external dissipation mechanism (formally, the role of a sink is played by the point $k = \infty$). Strong hydrodynamic turbulence of incompressible fluid also belongs to this type, since the energy-containing region for Kolmogorov spectrum is at large k, too [2]. That is why the explosion regime of forming the spectrum is also possible in that type of turbulence [18]. In hydrodynamics, consideration of explosive regimes is the

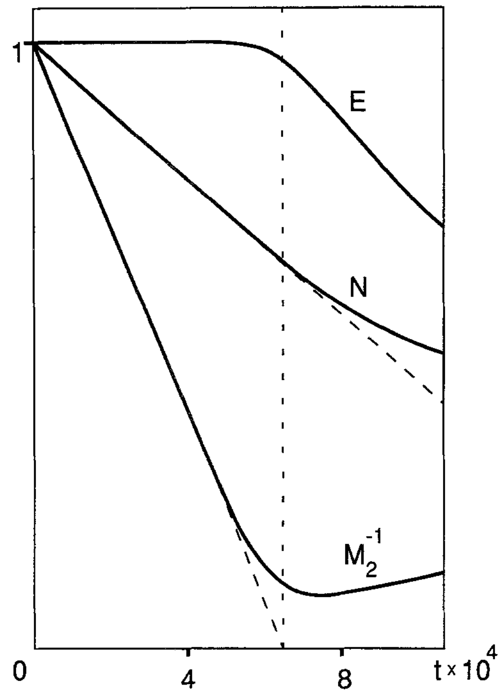

Fig. 2. Time dependence of all three moments of the distribution function for Var. 2, L = 1000. The vertical dashed line marks the time of the explosion.

way to resolve the D'Alembert paradox about the formal absence of drag in inviscid fluid: dissipation is still finite as viscosity tends to zero.

Despite the fact that the intermediate case $h = 0$, is degenerate and has zero measure in the space of conceivable wave systems, it is worth studying for the sake of completeness. This is especially true because two quite real physical systems, the gravitational-capillary and capillary waves on shallow water, are of this type. Normally, in the degenerate points separating the regions of different behavior (such as the regions $h < 0$ and $h > 0$), there is a peculiar type of self-similarity. Indeed, the condition $h = 0$, according to (10,11), implies independence of the characteristic interaction time of the wave number. Therefore the relaxation-front velocity in the logarithmic scale should be constant, which requires the self-similarity to be of exponential rather than power type:

$$n(k, t) = e^{-qt} f(ke^{-pt}) = e^{-qt} f(\xi). \qquad (22)$$

For (22) to be a solution of (14), one should impose the condition $q = p(2m + d - \alpha) = p(m + d) = ps_0$. In this case the energy

$$E = e^{(ps_0 - q)t} \int_0^\infty f(\xi)\xi^{\alpha+d-1}\, d\xi = \int_0^\infty f(\xi)\xi^{\alpha+d-1}\, d\xi$$

remains unchanged, therefore such a self-similar solution corresponds to the free evolution regime. After the self-similar front (22) has reached the sink, the evolution may switch over to the regime (19,20) with the energy diminishing in inverse proportion to time (see Eq. 19 and Fig. 2).

Thus, according to the character of evolution, the wave systems with the decay dispersion law are subdivided into two classes: one with $\alpha > m$ $(h > 0)$ and the other with $\alpha < m$ $(h < 0)$.

In the nondecay case, the classification also involves two Kolmogorov solutions:

1. The solution $n_1(k) \propto k^{-s_1}$, $s_1 = d + 2m/3$, with the energy flux toward the short-wave region; and
2. The solution $n_2(k) \propto k^{-s_2}$, $s_2 = d + 2m/3 - \alpha/3$, with the wave action flux to the long-wave region.

Accordingly, there are two important exponents: h_1 and h_2, which determine the location of the energy-containing region in the n_1 solution and the location of the region where the main part of wave action of the n_2 solution is concentrated:

$$E = \int \omega_k n_1(k)\, d\mathbf{k} \propto k^{\alpha+d-s_1} = k^{\alpha-2m/3} \equiv k^{h_1}, \tag{23}$$

$$N = \int n_2(k)\, d\mathbf{k} \propto k^{d-s_2} = k^{(\alpha-2m)/3} \equiv k^{h_2}. \tag{24}$$

So long as we deal with systems where $\alpha > 0$, we have $h_1 > h_2$. Consequently, there are three domains in the parametric space of systems:

1. $h_1 > 0$ and $h_2 > 0$;
2. $h_1 < 0$ and $h_2 < 0$;
3. $h_1 > 0 > h_2$,

which differ in the character of evolution of weakly turbulent distributions. This classification and analysis of nonexplosion regimes was given by Zakharov [15] earlier.

In the presence of a source, the Kolmogorov solution $n_1(k)$ at $k \to \infty$ is established by a self-similar front of the form (18) for $h_1 > 0$ and an explosion front (13) for $h_1 < 0$, whereas the $n_2(k)$ solution at $k \to 0$ is established in a quite opposite way, by type (13) for $h_2 < 0$ and (18) for $h_2 > 0$. In both of these cases, $p = 1/h_i$, $i = 1, 2$.

We shall briefly outline different cases of the free evolution of turbulence distributions. In domain 1 the long-wave Kolmogorov asymptotics corresponding to the constant flux of action forms explosively: $n(k,t) = (t_0 - t)^q f[k(t_0 - t)^{-p}]$, $f(x) \propto x^{-s_2}$ at $x \gg 1$, $p = 1/h_2$ and $q = ps_2$. Then, at $t > t_0$, the self-similarity regime of type (13) is established, with $q = p(\alpha + d)$ and $p = (3h_1)^{-1}$ due to energy conservation. Such a distribution drifts with time to the short-wave region. Defining the position of the energy-containing region in the ω-space as $\omega_E = E/N$, we find that, since E is conserved while N diminishes, the energy is pumped over into the large-ω region. In domain 2 the short-wave Kolmogorov asymptotics corresponding to a constant energy flux evolves by the explosion law (18). Then a front slowly forms, which propagates into the long-wave region by the self-similar law (13) with

$q = pd$ and $p = (3h_2)^{-1}$ corresponding to conservation of the wave action integral. The energy distribution maximum ω_E decreases with time. Finally, in domain 3 for both of the integrals E and N must be conserved, which precludes the existence of two-parameter self-similar solutions like (13) or (18). Therefore the evolution is, apparently, non-self-similar.

For example, in the case of wind-induced gravitational waves on deep water, $\alpha = \frac{1}{2}$ and $m = 3$, so we come to case (b), since $h_1 = -\frac{3}{2} < 0$ and $h_2 = -\frac{11}{6} < 0$. Thus, the Kolmogorov spectrum with an energy flux is formed at the short-wave region explosively: $n(k,t) = (t_0 - t)^{8/3} f[k(t_0 - t)^{2/3}]$. The long-wave spectrum with an action flux is formed by the decelerating relaxation front: $n(k,t) = t^{23/11} f(kt^{6/11})$. The boundary frequency of such a spectrum moves according to the law $\omega \propto t^{-3/11}$. Such a behavior is in qualitative agreement with the observations reported in [16,17], although detailed comparison (especially regarding the small-scale explosion regime) seems unrealistic so far, mainly due to anisotropic character of the wave ensembles observed and the incompleteness of data.

How is the attenuation of waves effected after the wind abates? The distribution has the Kolmogorov asymptotics with a constant energy flux at $k \to \infty$, so the energy does not have to be conserved. The evolution of decaying turbulence should come to the self-similar regime $n(k,t) = t^{4/11} f(kt^{2/11})$. Thus, in the isotropic case, the mean frequency of waves decreases: $\omega_E \propto t^{-1/11}$.

6. Method of Moments

We hope the reader realizes that analysis in the previous section was done at the level of plausible reasoning rather than at that of strict proofs. Indeed, not only did we have to take it for granted that an arbitrary distribution comes into the self-similar regimes (13) or (7), but also the very existence of self-similar solutions was not given any rigorous substantiation.

However, it turns out that the most interesting property of the evolution, the explosion character of the Kolmogorov asymptotics formation, may be given a rigorous mathematical proof for a particular case of weak acoustic turbulence. The proof is based on the analysis of moments of the distribution $n(k,t)$:

$$M_i(t) = \int k^i n(k,t)\, d\mathbf{k}.$$

If at $k \to \infty$ the power asymptotics $n(k) \propto k^{-s}$ is established for a finite period of time, then the moments M_i for $i > s - d$ must at some instant become infinite. The evolution of three-dimensional isotropic distributions of weak acoustic turbulence is described by the following equation in dimensionless variables:

$$\frac{\partial n(k)}{4\pi \partial t} = \int_0^k (k - k_1)^2 [n(k_1)n(k - k_1) - n(k)n(k_1) - n(k)n(k - k_1)] k_1^2\, dk_1$$
$$-2\int_k^\infty (k_1 - k)^2 [n(k)n(k_1 - k) - n(k_1)n(k_1 - k) - n(k_1)n(k)] k_1^2\, dk_1. \qquad (25)$$

As we consider the weak dispersion limit $\omega_k \approx k$, the energy is the first moment of the distribution function

$$E = M_1 = \int kn(k,t)\,d\mathbf{k} = 4\pi \int_0^\infty k^3 n(k,t)\,dk,$$

which is conserved if function $n(k,t)$ diminishes faster than by the Kolmogorov law $n(k) \propto k^{-9/2}$ at $k \to \infty$. For the case in question, $\alpha = 1$, $m = \frac{3}{2}$ and $h = -\frac{1}{2}$.

Let us consider other moments of the distribution function. For zeroth moment, which is the total number of waves $N(t) = 4\pi \int_0^\infty n(k,t)k^2\,dk$, we compute from (25)

$$\begin{aligned}\frac{dN}{dt} &= -(4\pi)^2 \int_0^\infty k^2\,dk \int_k^\infty k_1^2 (k-k_1)^2 \\ &\quad \times [n(k)n(k_1-k) - n(k_1)n(k) - n(k_1)n(k-k_1)]\,dk_1 \\ &= (4\pi)^2 \int_0^\infty k^2 n(k)\,dk \int_0^\infty k_1^2 n(k_1)\big[(k-k_1)^2 - (k+k_1)^2\big]\,dk_1 \\ &= -4E^2. \end{aligned} \tag{26}$$

In deriving this equation, we used a rearrangement of integration limits, which is admissible provided the integral $\int_0^\infty k^3 n(k)\,dk$ converges. Thus, (26) is valid if $n(k)$ falls off more rapidly at $k \to \infty$ than k^4 does. Note that we assume no singularities of $n(k)$ at $k = 0$. It follows from (26) alone that a distribution with finite and nonzero N and E that diminishes quickly at $k \to \infty$ initially, cannot retain a rapidly falling tail forever.

Equation (26) prompts the following scenario of evolution: during the time $t_0 = N(0)/4E^2$, the initially long-wave packet spreads throughout the k-space, N becomes zero, and the energy-containing scale $k_E = E/N$ becomes infinite. This scenario corresponds to the pumping-over of all the energy to infinity (actually to large k) for a finite period of time.

Actually, the evolution does not follow this scenario.

Let us consider the second moment $M_2 = 4\pi \int_0^\infty k^2 n(k)k^2\,dk$:

$$\begin{aligned}\frac{dM_2}{dt} &= 6M_2^2 + 8\pi E \int_0^\infty k^3 n(k)k^2\,dk \\ &\quad + 32\pi^2 \int_0^\infty u^2 n(u)\,du \int_0^u v^4 (u-v)^2 n(v)\,dv \geq 8M_2^2. \end{aligned} \tag{27}$$

Here we have used a simple inequality

$$\int_0^\infty k^5 n(k)\,dk \int_0^\infty k^3 n(k)\,dk \geq \left(\int_0^\infty k^4 n(k)\,dk\right)^2.$$

The inequality (27) is valid provided that the integral

$$M \propto \int_0^\infty k^4 n(k)\, dk$$

converges, that is, only if $n(k)$ diminishes faster than k^{-5}.

The equation $dx/dt = 8x^2$ has an explosion solution $x(t) = x(0)/\big(1 - 8x(0)t\big)$. Consequently, (27) demands that M_2 become infinite by the time $t \leq t_1 = [8M_2(0)]^{-1}$. It should be noted that the same method of moments could also be applied to the proof that the blow-up exists in some model system suggested for describing hydrodynamic turbulence [18].

It is easy to show that $t_1^{-1} = 8M_2(0) \geq 2t_0^{-1} = 8E^2/N(0)$, that is, that the explosion associated with the second moment M_2 becoming infinite occurs at least twice earlier than N becomes zero. We shall see below (see also Figs. 1 and 2) that M_2 increases with the Kolmogorov asymptotics $k^{-9/2}$ explosively forming in the large-k region.

In the distribution $n(k) \propto k^{-9/2}$, the energy-containing region is that of small k: $\varepsilon \propto k^{-3/2}$. Therefore, only a small part of the initial energy of turbulence appears in the region of large k until the Kolmogorov asymptotics has established itself. The effect of explosive formation of the Kolmogorov distribution in the short-wave region may be compared with a weak collapse (see [14]) when the value of an integral of motion captured into the singularity (in our case, into $k \to \infty$) tends to zero.

Thus, the evolution of a weak acoustic turbulence distribution will be two-stage. During the first (explosive) stage, energy is conserved, the number of waves decreases by a linear law, and the second moment increases explosively. As a result, a finite energy flux sets on at $k \to \infty$ to cause a decrease in the total energy. Then at $t \gg t_1$, the system comes into a type (13) self-similar regime with $p = 0$: $n(k, t) = t^{-1} f(k)$, with the energy decreasing by a power law: $E \propto t^{-1}$. The short-wave asymptotics in this case is of Kolmogorov type: $n(k) \propto k^{-9/2}$, so that (26) remains valid and the full number of waves monotonically decreases (this time not by the linear law though, see Fig. 2).

Note that, as the averaged interaction coefficient in (25) is proportional to k^2, consideration of any moments higher than the second one will change nothing in this scenario: their derivatives can be expressed in terms of the lower-order moments.

One can see from (25) that the reverse time of nonlinear interaction $t_{\mathrm{NL}}^{-1} \propto kE$ (cf. Eq. 11) grows with k more slowly than the dispersion correction to frequency $\delta\omega = a^2k^3$. This means that with the distribution drifting towards large k, the applicability criterion of the weak turbulence approximation, $\delta\omega t_{\mathrm{NL}} \gg 1$, will be increasingly better satisfied. Consequently, the formation of the power Kolmogorov asymptotics $n(k) \propto k^{-9/2}$ in an infinite k-space will proceed up to the absorption region or to $k \simeq a^{-1}$ where the dispersion will cease being small and (25) will become inapplicable.

Also note that the presence of an external wave source, that is, the addition of positive terms like F_k (external force) or $\gamma_k n_k$ (increment) into the right-hand side of (25) would not violate (27). Therefore, forming the stationary Kolmogorov spectrum will take a finite period of time as well.

7. Numerical Simulation

Any real experiment, be it natural or numerical, deals with a finite number of modes. This is due to either finiteness of the system observed and discreteness of the medium, or strong attenuation of high harmonics. Accordingly, we carried out two variants of numerical simulation of (25):

Var. 1: *for a closed system of L modes,*

$$\begin{aligned}\frac{\partial n(k,t)}{\partial t} &= \sum_{l=1}^{k} l^2(k-l)^2\{n(l)n(k-l) - n(k)[n(l) + n(k-l)]\}\\ &\quad - 2\sum_{l=k}^{L} l^2(l-k)^2\{n(k)n(l-k) - n(l)[n(k) + n(l-k)]\}\\ &= W_k; \end{aligned} \tag{28}$$

Var. 2: *for an open system with the waves going out (we set $n(k) \equiv 0$ at $k > L$),*

$$\frac{\partial n(k,t)}{\partial t} = W_k - 2n(k)\sum_{l=L}^{L+k} l^2(l-k)^2 n(l-k). \tag{29}$$

The last term in (29) corresponds to sweeping waves toward the region $k > L$ due to confluence processes. It plays the role of nonlinear damping and provides for effective dissipation of energy. The evolution in the closed system must lead to forming the equilibrium distribution $n(k) = T/k$ at $k \gg 1$, with conserved in (28). Total energy is written

$$E = \sum_{k=1}^{L} k^3 n(k).$$

For the total number of waves $N = \sum_{k=1}^{L} k^2 n(k)$, we have from (28):

$$\frac{\partial N}{\partial t} = -4E^2 + \sum_{k=1}^{L} k^2 n(k) \sum_{l=L-k}^{L} l^2(k+l)^2 n(l).$$

The last term has the meaning of the "overlapping integral" over the regions $(1, L/2)$, $(L/2, L)$ and, consequently, as the waves are distributed over the whole interval, the law of N decrease will increasingly deviate from the linear one.

In the open system (29), the energy monotonically diminishes

$$\frac{\partial E}{\partial t} = -2\sum_{k=1}^{L} k^3 n(k) \sum_{L-k}^{L} l^2(k-l)^2 n(l),$$

also owing to a similar "overlapping integral."

Thus if a wave packet was placed initially in the region $k \ll L$, the evolution in both cases (28) and (29) will proceed identically until finiteness of the k-space manifests itself. In the closed system, the waves will gather near the right edge.

In the open one, on the contrary, the occupation numbers will fade away (in contrast to the evolution in an infinite system), because of nonlinear damping. In the moments of time and in the regions of the k-space where the solutions (28) and (29) are close, one can deem the numerical experiment a good simulation of wave behavior in an infinite medium.

The initial distribution was chosen to be localized in the region of small k

$$n(k,0) = \exp[-k^2/k_0^2] = \exp[-k^2/10].$$

Equations (28,29) were solved numerically, the time derivative being approximated by the first difference

$$\frac{\partial n(k,t)}{\partial t} \approx \frac{n(k,t+\Delta t) - n(k,t)}{\Delta t},$$

where Δt was made sufficiently small ($5 \cdot 10^{-5} \div 2 \cdot 10^{-8}$) to provide for stability of the numerical procedure. The number of modes was chosen to be equal to 200, 400 or 1,000.

Figure 1 presents the evolution of the $n(k,t)$ value up to the moment in time at which the solutions (28) and (29) start diverging. As we see, by the time $t^* = 5.3 \cdot 10^{-4}$, the Kolmogorov distribution transferring the constant energy flux to large k spans almost the whole of the interval of 1,000 points. The details of this are depicted in Fig. 3, which shows the dependence of the local exponent of the spectrum on k

$$s(k) \equiv -\frac{\ln\bigl(n(k)/n(k+1)\bigr)}{\ln(k/k+1)}.$$

It is seen that noticeable difference in the behavior of the open and closed systems is observed from the time $t \simeq t^*$. Then in the closed system the distribution quickly becomes less steep ($s(k)$ decreases), and an equilibrium spectrum forms with $s(k) \equiv 1$, starting from the right end of the interval.

Now let us look again at Fig. 2. One can see that there are two clear-cut stages of evolution. Approximating the dependence $M_2^{-1}(t)$ by a dashed line, we get $t^* = 5.32 \cdot 10^{-4} \approx [8.7M(0)]^{-1}$. The fact that the $M_2^{-1}(t)$ value decreases without reaching zero and that the energy starts going down earlier than at $t = t^*$ is, of course, due to finiteness of the system, see Fig. 4.

Let us now check the supposition formulated in Sect. 5 that the self-accelerated process of Kolmogorov asymptotics formation occurs in the region $k \gg k_0$ at $\tau = t^* - t \ll t^*$. The equation (25) admits a self-similar explosion-type substitution

$$n(k,t) = \tau^{-5p-1} f(k\tau^{-p}) = \tau^{-5p-1} f(z). \tag{30}$$

One can expect formation of self-similar asymptotics under the condition that the occupation numbers change much quicker in the region $k \gg k_0$ than at $k \simeq k_0$. Then matching the self-similar part of the solution with the energy-containing region $k \simeq k_0$ should be fulfilled via a quasistationary intermediate asymptotics. This asymptotics can have the Kolmogorov type only if $f \propto z^{-9/2}$ at $z \ll 1$. So for a steady state $p = -2$. The region containing the Kolmogorov distribution should be bounded by $z \simeq 1$ and should expand to infinity for a finite time: $k_b \propto (t^* - t)^{-2}$. We computed

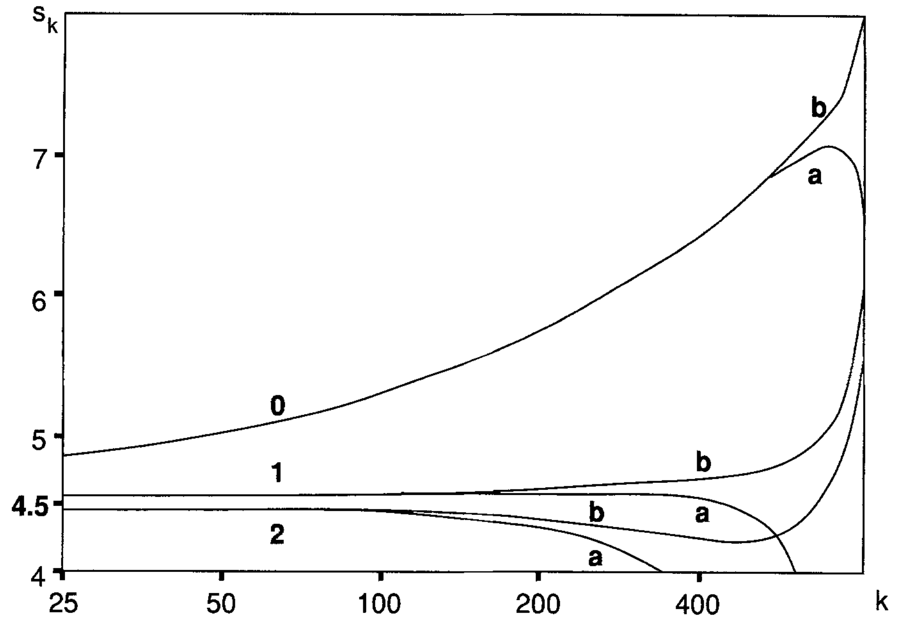

Fig. 3. Dependence of the local exponent of the spectrum on k at different moments in time. 0: $t = 4 \cdot 10^{-3}$, 1: $t = 5.3 \cdot 10^{-4}$, 2: $t = 6 \cdot 10^{-4}$. The "a" and "b" markers imply that the corresponding curves were obtained for Var. 1 and Var. 2, respectively.

the function

$$p(k, t) = \frac{(t^* - t)[\partial \ln n(k)/\partial t] - 1}{[\partial \ln n(k)/\partial \ln k] + 5},$$

which, for a solution of the form (30), must depend neither on k nor on t. At least qualitatively, our numerical simulation supports this (see Fig. 5) . We shall say, however, that observation of the self-similar mode (30) in a finite system is rather difficult, since at t close to t^* the asymptotics at large k approaches the Kolmogorov assymptotics, and the influence of the right boundary of the interval becomes apparent. For this reason, the dimension of the region where p is approximately constant decreases with time because of the distribution drift to large k and the increasing influence of the dumping.

In agreement with what was said above the maximal p decreases approximately down to $p \approx 2$, see Fig. 5. At $t \gg t^*$, the law of energy E decrease approaches the power law $E \propto t^{-\beta}$, which may be seen in Fig. 6 showing the function

$$\beta(t) = \frac{\partial \ln E}{\partial \ln t}.$$

According to our supposition, $\beta(t)$ must tend to unity with increase in time. Figure 6 is consistent with this supposition.

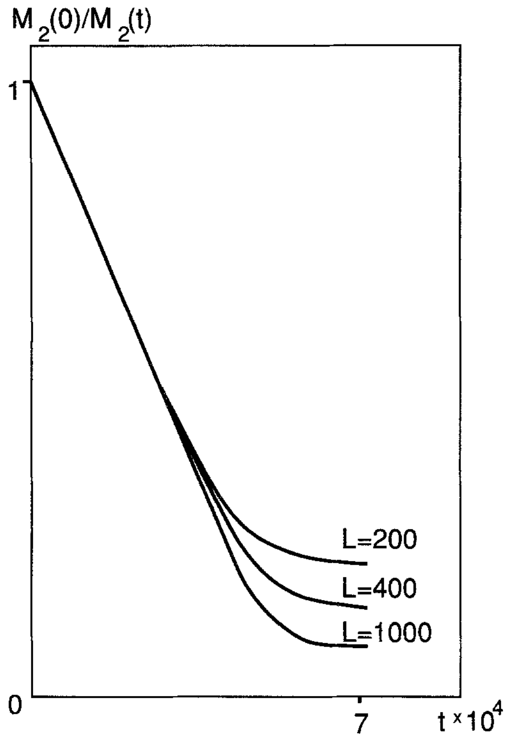

Fig. 4. Functions $M_2^{-1}(t)$ for a different number of modes L.

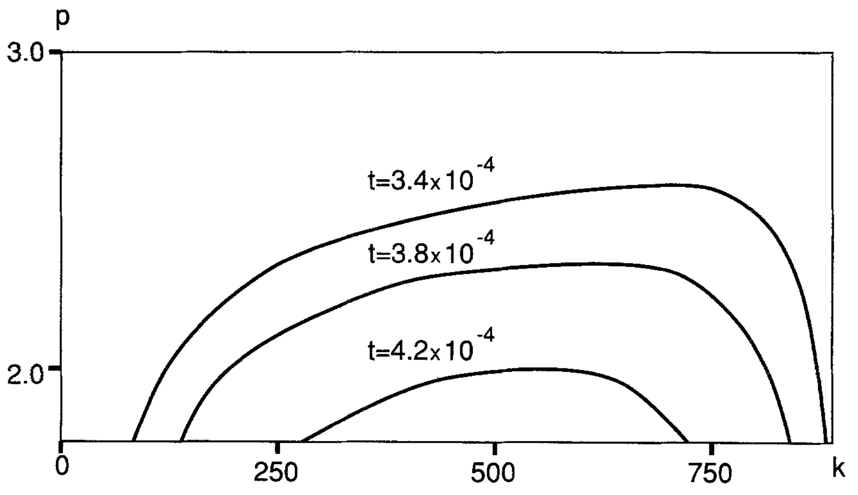

Fig. 5. Exponent $p(k, t)$ for $t^* = 5.3 \cdot 10^{-4}$, $L = 1000$.

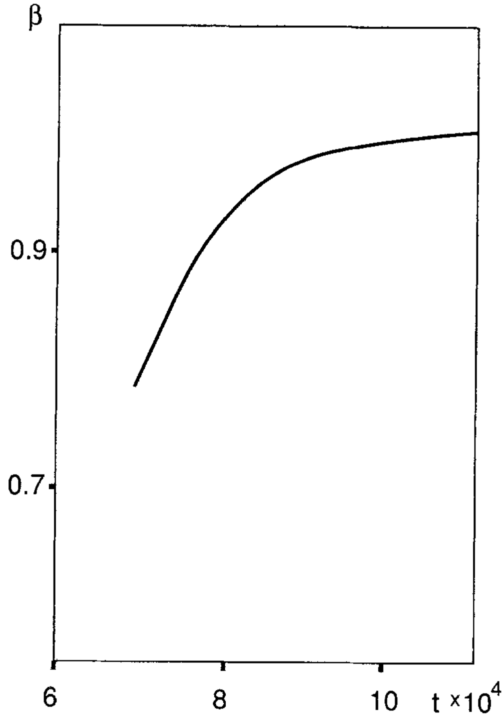

Fig. 6. Energy exponent β versus time.

Concluding this paper, we shall consider two-dimensional weak acoustic turbulence at $d = 2$ presenting an example of a system with $h = 0$. The isotropic kinetic equation describing that type of turbulence is similar to (25):

$$\begin{aligned}\frac{\partial n(k,t)}{2\pi\partial t} &= 2\int_0^k (k-k_1)[n(k_1)n(k-k_1) - n(k)n(k_1) - n(k)n(k-k_1)]k_1\,dk_1 \\ &\quad - 2\int_k^\infty (k_1-k)[n(k)n(k_1-k) - n(k_1)n(k_1-k) \\ &\quad - n(k_1)n(k)]k_1 dk_1. \qquad (31)\end{aligned}$$

Since the averaged interaction coefficient is proportional to the first power of k, it is reasonable to consider the behavior of two moments of the distribution function $n(k)$: the first and zeroth ones. The first moment $\int_0^\infty kn(k)k\,dk = E$ is the energy which is conserved when external damping is absent. As far as the zeroth moment (i.e., the total number of waves) is concerned one can obtain from (31):

$$\begin{aligned}\frac{dN}{dt} &= 2\int_0^\infty kn(k)\,dk\int_0^\infty k'(k+k')n(k+k')\,dk' - 2NE \\ &= -\int_0^\infty dk\left(\int_k^\infty k'n(k')\,dk'\right)^2.\end{aligned}$$

On the one hand, one can see that $dN/dt \leq 0$, that is, the energy-containing scale $k_E = E/N$ of an arbitrary distribution drifts monotonically to large k. On the other hand, $dN/dt \geq -2NE$, that is, $N(t)$ does not decrease faster than by the exponential law. We had examined the evolution of $n(k, t)$ in a discrete open system corresponding to (31) until the time that the energy decreased almost by a factor of three. From Fig. 7 one can see that there are two stages of evolution. During the first stage energy is conserved and the Kolmogorov distribution forms up to the dissipation region (in the given case $n(k) \propto k^{-3}$).

The time t_1 of forming the constant energy dissipation rate grows approximately logarithmically with the size of the system: $L = 50$, $t_1 \simeq 5.70$; $L = 100$, $t_1 \simeq 8.05$; $L = 200$, $t_1 \simeq 10.70$; $L = 400$, $t_1 \simeq 13.49$, which is in agreement with (22). At $t \gg t_1$ the self-similar regime (13) sets in, and the energy decreases as t^{-1}. Such a regime persists until the time that the energy concentrated in the region unaffected by external damping (approximately $k \leq L/2$) becomes comparable to the energy in the interval $k \geq L/2$. After this, the energy dissipation rate starts to decrease.

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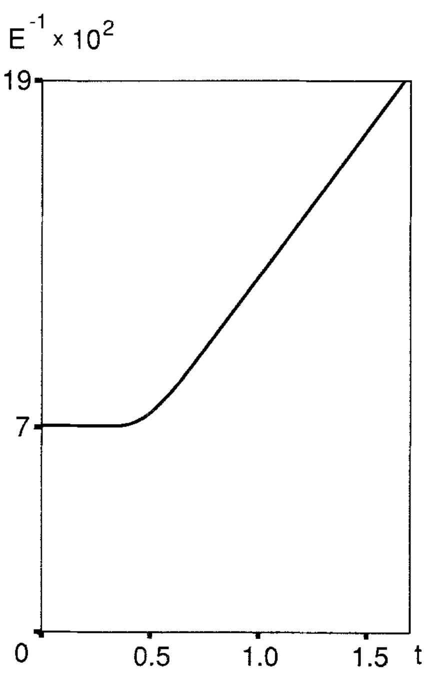

Fig. 7. Energy versus time for two-dimensional acoustic turbulence with $L = 400$.

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