Kolmogorov spectra of Langmuir and optical turbulence

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Weak developed turbulence in the framework of both scalar and vector nonlinear Schrödinger equations is considered. It corresponds to waves with a quadratic dispersion law $\omega_k = \omega_0 + \beta k^2$ and with a zero scaling exponent of the interaction coefficient. The consideration thus embraces the turbulence of envelopes (usually referred to as optical turbulence) as well as Langmuir turbulence in nonisothermal plasma and other examples. Steady spectra of turbulence are shown to be close to Kolmogorov-like cascade spectra with the fluxes of energy and wave action.

I. INTRODUCTION

The problem of turbulence has recently been recognized as one having significance for the whole of physics rather than for hydrodynamics only. Following Ref. 1, we define developed turbulence as a highly excited chaotic state of a continuous medium with a large number of degrees of freedom being deviated from the thermodynamic equilibrium. In hydrodynamic language, this means that the Reynolds number is extremely large. It could be obtained if the scales of pumped motions and of those that are effectively damped are very different. Nonlinear interaction provides a redistribution of energy (and of other motion integrals) between different modes. The main task of the theory is thus to find a stationary law of energy distribution that is a turbulence spectrum. One may expect this law to be universal for scales intermediate between those of pump and sink. The range of such intermediate scales is usually called an inertial interval.

The basic physical idea here is one put forward by Richardson, Kolmogorov, and Obukhov[1-3] about relay energy transfer in the inertial interval. Assuming interaction to be local, one can conclude that a steady spectrum should carry a constant flux of a motion integral. The spectrum could thus be guessed from the dimensional analysis. Such a cascade picture has proven to be especially useful in considering weak turbulence, which consists of the weakly interacting waves. In that case, the description of turbulence is provided in terms of double correlators, which obey a close kinetic equation. A universality hypothesis turns into a strict theory in this case: locality could be proven directly and cascade spectra can be obtained as exact solutions of the kinetic equation (see Ref. 1).

In this paper, we consider three-dimensional weak turbulence in the framework of two universal wave models: the nonlinear Schrödinger equation and Zakharov's equations. The first model looks in dimensionless variables as follows:

$$i\psi_t + \Delta\psi + T|\psi|^2\psi = 0. \qquad (1)$$

It has quite a wide scope of applications. First of all, it describes the behavior of a narrow packet envelope for most nonlinear wave systems of general position. In particular, it governs the behavior of light in nonlinear dielectrics, which is the reason that the turbulence of envelopes is usually referred to as optical turbulence. The nonlinear Schrödinger equation also describes the quasiclassical limit of a weakly nonhomogeneous Bose gas (see Refs. 4 and 5) and long spin waves in antiferromagnets and many other systems with the quadratic dispersion law and with the coefficient of nonlinear interaction being approximately constant. The second model, viz., Zakharov's equations, has about the same degree of universality. It has been obtained originally by Zakharov[6] for Langmuir waves interacting with an ion sound in nonisothermal plasma. In fact, those equations describe any physical system that contains two types of mutually interacting waves: high-frequency waves with the quadratic dispersion law and a low-frequency sound. We consider Zakharov's system in the so-called static approximation (see Ref. 6) when it turns into the vector nonlinear Schrödinger equation:

$$\nabla\cdot(i\mathbf{E}_t + \Delta\mathbf{E}) + T\nabla\cdot(|\mathbf{E}|^2\mathbf{E}) = 0. \qquad (2)$$

From the viewpoint of weak turbulence theory, the above equations are very similar since the scaling exponents of dispersion ($\alpha = 2$) and that of a four-wave interaction coefficient ($m = 0$) are the same for them. Of course, there is quite a narrow range of external parameters for the existence of purely weak turbulence (especially in the plasma). Strongly nonlinear phenomena like self-focusing and plasma collapse may occur. However, strong and weak turbulence could coexist taking place in different regions of k space as it is described, for instance, in Ref. 7. Collapses provide a large-scale sink for the weakly turbulent inverse cascade (see Refs. 6 and 7 for details). Below we discuss transition of a weak Kolmogorov-like turbulence into a strong one in more detail. Here we would like to review briefly what is known about the steady spectra of a developed weak turbulence in the framework of a three-dimensional (3-D) nonlinear Schrödinger equation. The presence of two motion integrals: energy and action (number of waves), means that there should be two turbulent spectra: one that carries wave action from the pump to large scales (inverse cascade) and the

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other one that carries energy to small scales. The directions of the integral transfer could be established by simple speculations similar to Fjortoft's theorem in hydrodynamics (see, e.g., Refs. 1 and 2). Unfortunately, only the spectrum with the action flux was obtained in Ref. 6 as an exact stationary solution of wave kinetic equation. Moreover, numerical simulations of Hansen and Nicholson[8] demonstrated that the large-scale spectrum vastly differs from that suggested by Zakharov.[6] As far as the energy transfer is concerned, one could formally suggest from the dimensional analysis the Kolmogorov-like spectrum carrying constant energy flux, but the spectrum is proven to be nonlocal (see Ref. 6). Nonlocality means that such a spectrum is not a solution of the kinetic equation; nor does it correspond to the cascade hypothesis.

So far, there is thus no clear understanding of how the motion integrals are transported from the pump to the damping regions, even if the pump is a weakly turbulent one. Our paper is devoted to this matter. Here we demonstrate that both motion integrals are transferred by cascade processes. Namely, we found in numerical simulations that the large-scale spectrum is close to that suggested by Zakharov and explain why a different spectrum was obtained in earlier numerical experiments. As far as the small-scale part of turbulence is concerned, we demonstrate it to be slightly deviated (by a logarithmic factor) from a naive dimensional estimate. Such a deviation provides the constancy of the energy flux in the inertial interval. That agrees with the numerical experiments. Thus, both steady spectra of weak turbulence are close to the universal cascade spectra. It is true both for scalar and vector nonlinear Schrödinger equations.

II. ANALYTICAL TREATMENT

The universal language for weak turbulence theory is the Hamiltonian description in terms of the normal wave amplitudes (see, e.g., the first chapter of Ref. 1).

Both (1) and (2) could be written in the standard Hamiltonian form after the Fourier transform: $\psi(\mathbf{r},t)\rightarrow a(\mathbf{k},t)$ for the scalar equation (1) and $\mathbf{E}(\mathbf{r},t)=\nabla\phi(\mathbf{r},t)$, $a(\mathbf{k},t)=k\phi(\mathbf{k},t)$ for the vector one (2). As a result, we have

$$i\frac{\partial a(\mathbf{k},t)}{\partial t}=\frac{\delta H}{\delta a^*(\mathbf{k},t)},$$

$$H=\int\omega_k|a_k|^2\,d\mathbf{k}-\frac{1}{2}(2\pi)^{-3}\int T_{k123}a_k^*a_1^*a_2a_3 \times\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)d\mathbf{k}\,d\mathbf{k}_1\,d\mathbf{k}_2\,d\mathbf{k}_3. \quad (3)$$

Here the dispersion law is $\omega_k=k^2$, since we have excluded the frequency gap ω_0 from the very beginning (the only memory is the absence of three-wave interactions at $k^2<2\omega_0$). The coefficient of four-wave interaction is equal to the constant T for (1) and it has the following form for (2) (see Ref. 9):

$$T_{k123}=T\{[(\mathbf{k}\mathbf{k}_2)(\mathbf{k}_1\mathbf{k}_3)+(\mathbf{k}\mathbf{k}_3)(\mathbf{k}_1\mathbf{k}_2)]/kk_1k_2k_3\}. \quad (4)$$

The scaling exponent of the interaction coefficient is equal to zero.

We consider the case of weak nonlinearity when the first term in the Hamiltonian (responsible for linear phenomena) prevails over the second one describing wave interaction. In this case, averaging over random phases, one could obtain the kinetic equation for the double correlation function of the wave field $\langle a(\mathbf{k},t)a^*(\mathbf{k}',t)\rangle=n(\mathbf{k},t)\delta(\mathbf{k}-\mathbf{k}')$ (as it is explained in Refs. 1,8, and 10). Correlator n_k slowly varies only due to the nonlinear interaction:

$$\frac{\partial n_k}{\partial t}=4\pi\int|T_{k123}|^2\delta(\omega_k+\omega_1-\omega_2-\omega_3)\times\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)n_kn_1n_2n_3\times(n_k^{-1}+n_1^{-1}-n_2^{-1}-n_3^{-1})d\mathbf{k}_1\,d\mathbf{k}_2\,d\mathbf{k}_3. \quad (5)$$

From the physical viewpoint, the condition of a weak turbulence means that the wave frequency ω_k should be much larger than the typical inverse time of nonlinear interaction $t_{\rm nl}^{-1}$. The latter could be estimated from (5): $t_{\rm nl}^{-1}\simeq Tn_k^2k^6/\omega_k$. Therefore the applicability condition of below results is as follows:

$$\xi=Tn_k^2k^6/\omega_k^2\ll 1. \quad (6)$$

As one can see, the kinetic equation contains only $|T_{k123}|^2$, so the weak turbulence description is insensitive to the sign of the interaction coefficient. However, a strong turbulence depends on the sign of interaction crucially, since collapse could exist only for positive nonlinearity ($T>0$), which corresponds to mutual attraction of waves. As we demonstrate below, a weak turbulence turns into a strong one in the large-scale region. A difference between systems with the different signs of the nonlinearity should arise there.

Our aim is to study isotropic steady distributions. Supposing $n(\mathbf{k})=n(k)=n(\omega_k)=n(\omega)$, we can write (5) in ω space as in Refs. 6 and 8:

$$\frac{\partial n(\omega,t)}{\partial t}=I(\omega)=\int S(\omega,\omega_2,\omega_3)\times nn(\omega_2+\omega_3-\omega)n_2n_3\times[n^{-1}+n^{-1}(\omega_2+\omega_3-\omega)-n_2^{-1}-n_3^{-1}]d\omega_2\,d\omega_3. \quad (7)$$

Here $n=n(\omega)$, $n_i=n(\omega_i)$; to avoid misunderstanding, we would like to emphasize that they are the densities of the waves in $\mathbf{k}$ space yet considered as functions of frequency. Integration in (7) is done over the shadow region in Fig. 1. Kernel S is expressed by different ways in the regions I–IV. For the scalar equation (1) when $T_{k123}\equiv T$, it is as follows:

$$S(\omega,\omega_2,\omega_3)=\begin{cases}\sqrt{(\omega_2+\omega_3-\omega)/\omega}, & \text{for I;}\\ 1, & \text{for II;}\\ \sqrt{\omega_3/\omega}, & \text{for III;}\\ \sqrt{\omega_2/\omega}, & \text{for IV.}\end{cases} \quad (8)$$

For the vector case (2), function S should be multiplied by the dimensionless function, which is too bulky to be presented here. The reader can find the complete expression in Ref. 8.

The kinetic equation (7) has two integrals of motion. The first one is the energy,

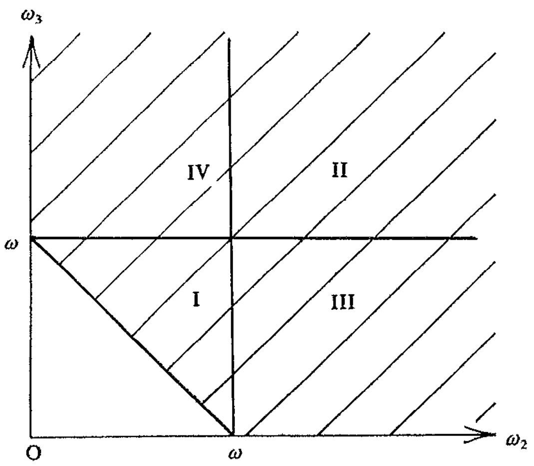

FIG. 1. Integration region for kinetic equation (7).

$$E=\int \omega_k n_k \, d\mathbf{k}=2\pi\int_0^\infty \omega^{3/2}n(\omega)d\omega,$$

while the second one is the wave action (that is, the total number of waves),

$$N=\int n_k \, d\mathbf{k}=2\pi\int_0^\infty \omega^{1/2}n(\omega)d\omega.$$

Any wave source (for example, a modulation instability) should produce both integrals. We consider a developed turbulence by supposing that the scales (and frequencies) of pumped waves are strongly different from those of damping waves. And it is quite clear that at least two damping regions should exist in ω space to provide a stationary distribution. Indeed, one can describe both the source and sink of waves by adding to the rhs of (7) some function $F(\omega)$, which might be a functional of $n(\omega)$ as well. The region of ω space with a positive sign of F plays the role of the pump, that with a negative sign corresponds to the damping region, and $F=0$ in the inertial intervals. In a stationary case, integrating (7) by virtue of the conservation laws, we obtain

$$\int_0^\infty F(\omega)\omega^{1/2}\,d\omega=0,\quad \int_0^\infty F(\omega)\omega^{3/2}\,d\omega=0.$$

From these two conditions, one can see that function $F(\omega)$ should be sign alternating and it should also change its sign at least twice. So the turbulence in question is supposed to be excited by the pump at some intermediate frequency ω_0, whereas two damping regions exist at some smaller ($\omega_1<\omega_0$) and larger ($\omega_2>\omega_0$) frequencies. Here we consider fully developed turbulence with two well-defined inertial intervals: $\omega_1 \ll \omega_0 \ll \omega_1$. One can readily find which motion integral is absorbed by each sink. To do this, let us assume that in the steady state the pump generates N_0 waves per unit time while the sinks absorb N_1 and N_2, respectively. Two conservation laws,

$$N_1+N_2=N_0,\quad \omega_1 N_1+\omega_2 N_2=\omega_0 N_0,$$

should be satisfied in the steady state. From this we obtain

$$N_1=N_0[(\omega_2-\omega_0)/(\omega_2-\omega_1)],$$
$$N_2=N_0[(\omega_0-\omega_1)/(\omega_2-\omega_1)].$$

It is seen that under the sufficiently large left inertial interval (as $\omega_1\ll\omega_0<\omega_2$) the whole energy is absorbed by the right sink: $\omega_2 N_2\approx\omega_0 N_0$. Similarly, at $\omega_2\gg\omega_0>\omega_1$, we have $N_1\approx N_0$, i.e., the wave action is absorbed at small ω. The above deduction is similar to that of Fjortoft's theorem in two-dimensional hydrodynamics (see Ref. 2 for details).

Thus we come to the problem of finding spectra of turbulence that carry motion integrals from source to sinks in the inertial intervals. The notions of fluxes should be introduced for this purpose. By virtue of the conservation laws, Eq. (7) could be written as a continuity equation either for wave action density $[N(\omega)=2\pi\omega^{1/2}n(\omega)]$,

$$\frac{\partial N(\omega,t)}{\partial t}+\frac{\partial Q(\omega)}{\partial\omega}=0,$$

or for energy density $[\epsilon(\omega)=2\pi\omega^{3/2}n(\omega)]$,

$$\frac{\partial \epsilon(\omega,t)}{\partial t}+\frac{\partial P(\omega)}{\partial\omega}=0.$$

Here we have introduced the fluxes of the energy and of the action. As one can see, a stationary distribution should correspond to the fluxes that are independent of frequency. Now it is very easy to obtain an expression for the Kolmogorov spectrum carrying a constant flux of a motion integral. For example, for the wave action, one can express the current flux in terms of the collision integral $I(\omega)$,

$$Q(\omega)=-2\pi\int_0^\omega \omega^{1/2}I(\omega)d\omega. \tag{9}$$

Requiring the flux to be constant, one could find that it corresponds to the power distribution

$$n(\omega)\propto Q^{1/3}\omega^{7/6}, \tag{10}$$

found by Zakharov.[6] One could verify that (10) turns the collision integral into zero. It could be checked either directly or using elegant Zakharov's transformations (see Refs. 6 and 1). Also it is easy to establish that the solution (10) is local, i.e., both the collision integral and the flux are defined by converging integrals in this case. Everything is fine, however, numerical simulations of Ref. 8 give the spectrum decreasing with ω substantially slower than (10) under high-frequency pump and low-frequency sink. We think such a deviation can be simply explained by the absence of a second sink in the numerical scheme of Ref. 8. As we have shown above, a steady state is impossible without a second damping region. So we suppose the distribution presented in Ref. 8 is nonstationary, though varying quite slowly. In addition, the absence of the high-frequency energy sink makes it impossible for the low-frequency spectrum to be defined by action flux only. In the next section, we show the results of numerical simulations of Eq. (7) with one source and two sinks. One can see that the exponent of the low-frequency spectrum is close to $\frac{7}{6}$ — see Fig. 2.

Now let us discuss the destiny of the energy produced by a pump. How could it reach a high-frequency damping region? Proceeding similarly to (9), we write down the expression for the current energy flux,

$$P(\omega)=-2\pi\int_0^\omega \omega^{3/2}I(\omega)d\omega. \tag{11}$$

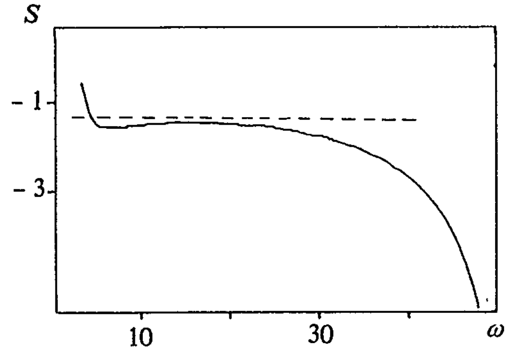

FIG. 2. Current exponent for steady distribution of Langmuir turbulence with the pump at $\omega - 48$ and two damping regions. The dashed straight line denotes the theoretical value of Kolmogorov exponent $s = -7/6$.

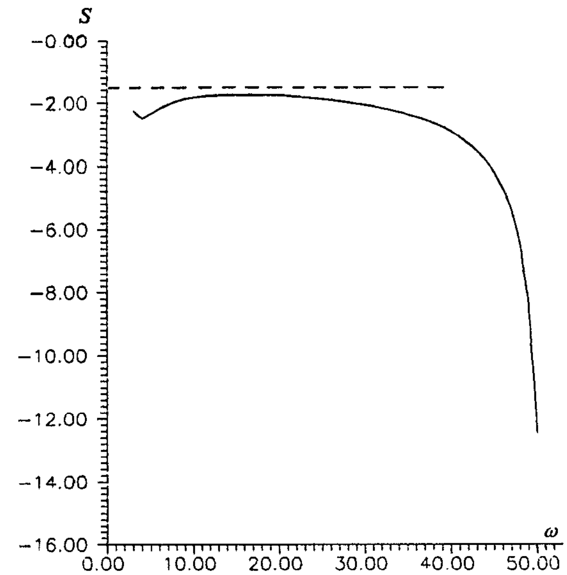

FIG. 3. Current exponent for steady distribution of Langmuir turbulence with the pump at $\omega = 2$ and two damping regions. The dashed straight line denotes the value $s = -3/2$.

Assuming power distribution $n(\omega) \propto \omega^s$, we find from (11) that $P(\omega) \propto \omega^{9/2+3s}$. One may try to require the flux to be independent of ω, which gives $s = -\frac{3}{2}$. However, substituting $n(\omega) \propto \omega^{-3/2}$ into (7) or (11) we find that the results diverge logarithmically. Indeed, the main term in $I(\omega)$ in this case corresponds to the integration over the regions of either small ω_3 or ω_2:

$$I(\omega) \propto \int_{\omega_0}^{\omega} \omega_3^{1/2} n(\omega_3)\,d\omega_3 \int_{\omega}^{\infty} n(\omega_2)\,d\omega_2 \propto \ln(\omega/\omega_0). \qquad (12)$$

Here we have introduced the pump frequency ω_0 as an integration cut off to avoid an integral divergency. Such a divergency means the absence of a simple power local solution that is expressed only in terms of flux value and frequency. However, we would like to draw the reader's attention to the fact that the divergency is weak (logarithmic). It allows us to look for the steady spectrum as slightly deviated from the "naive" estimate $\omega^{-3/2}$ at $\omega \gg \omega_0$. To provide a constancy of the energy flux in the main order in the large parameter $\ln(\omega/\omega_0)$, we substitute (12) into (11) with $n(\omega) \propto \omega^{-3/2}\phi(\omega/\omega_0)$. Assuming ϕ to be a slow function, we find $\phi = \ln^{-2/3}(\omega/\omega_0)$. Finally, the steady spectrum at high frequencies is equal,

$$n(\omega) = aP^{1/3}\omega^{-3/2}\ln^{-2/3}(\omega/\omega_0), \qquad (13)$$

a being dimensionless constant of order unity. Spectrum drops with ω a bit faster than by power law to compensate a small nonlocal gain of energy by the waves in the inertial range directly from the pump. Strictly speaking, the spectrum found is not universal since it is directly expressed via the pump frequency (in addition to the flux value and the current frequency). Yet such nonuniversality (and nonlocality) is weak. Indeed, calculating the current spectrum exponent, which is the logarithmic derivative,

$$s(\omega) = \frac{\partial \ln n(\omega)}{\partial \ln \omega} = -\frac{3}{2} - \frac{2}{3\ln(\omega/\omega_0)}, \qquad (14)$$

we see it to be close to $-\frac{3}{2}$. This is supported by our numerical simulations — see Fig. 3 below.

In conclusion of this section, let us discuss the applicability conditions of the solutions found. The dimensionless parameter of nonlinearity ξ is proportional to $(kn_k)^2$ according to (6). Since both spectra decrease faster than k^{-1}, the turbulence becomes stronger in the limit of large scales (which just corresponds to collapse or self-focusing).

III. NUMERICAL SIMULATIONS

We solved the kinetic equation (7) numerically for ω being an integer and changing from 1 to 50. On the rhs, we added the term $\Gamma(\omega)n(\omega)$ with positive Γ, which simulates the pump due to an instability. To provide necessary sinks, we used the zero boundary condition instead of imposing linear damping. We put $n(1) = 0$ and $n(\omega) \equiv 0$ for $\omega > 50$. Such a method was introduced by Zakharov and Musher[11] and it is discussed in detail in Refs. 1 and 12. Zero conditions beyond some interval provide a necessary outflux of waves, which allows better modeling of a developed turbulence than by using a linear damping, which usually leads to an extra accumulation of waves near the edges, as has been demonstrated in Ref. 13.

The calculations were performed both for optical and Langmuir turbulence, i.e., for scalar and vector cases, respectively. We used different initial distributions $n(\omega,0)$. The final spectrum was independent of the initial one in all cases. To check the numerical scheme, we started from the close (undriven undamped) system and made sure that it went to a steady equilibrium distribution $n(\omega) \propto (\omega + \mu)^{-1}$. Then we imposed zero boundary conditions and the pump. Shifting the pump along the ω interval, we could observe either a low- or high-frequency part of the spectrum to be more pronounced. We used slightly different forms of the growth rate $\Gamma(\omega)$ that had no influence on the saturated spectrum. All figures presented below correspond to $G_{max} = 1$ and an initial distribution was chosen as $n(\omega) \equiv 0.1$. The distribution is presented for $t = 4$, a rerun of

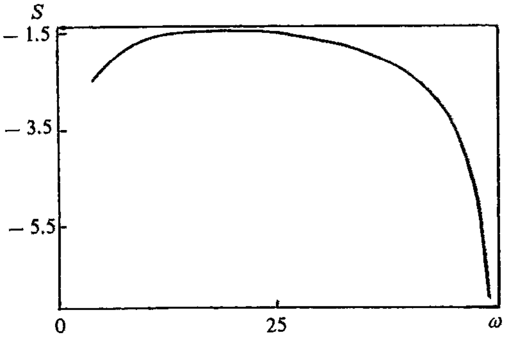

FIG. 4. Current exponent for steady distribution of optical turbulence with the pump at $\omega = 2$ and two damping regions.

the calculation procedure for the next unit of time changes the values of $s(\omega)$ less than by 0.001 in all cases.

First, we placed the maximum of $\Gamma(\omega)$ near the right end of our interval (at $\omega = 48$). We present in Fig. 2 the current exponent $s(\omega)$ calculated as a logarithmic derivative of final distribution according to formula (14). Such a form of data presentation allows us to resolve subtleties of spectrum behavior. Figure 2 corresponds to optical turbulence (with the interaction coefficient being a constant). The dashed straight line denotes the theoretical value of exponent $s = -\frac{7}{6}$ predicted by Zakharov for the cascade spectrum carrying a constant flux of the wave action. As one can see, while going deep into the inertial interval (from right to left), the observed value approaches the theoretical one. A sharp increase of the exponent at $\omega < 5$ means that the spectrum becomes less steep while approaching the damping region. Numerical simulations with the interaction coefficient (4) gave qualitatively the same picture.

Second, we put the pump near the left end of our interval, at $\omega = 2$. Figure 3 presents the steady values of the exponent for Langmuir turbulence. The dashed straight line denotes the value of the "naively" estimated exponent. The distribution obtained thus agrees well with our predictions [(13) and (14)]. The simulations for optical turbulence demonstrated the similar behavior presented in Fig. 4. Therefore, we can conclude that the high-frequency spectrum is close to the cascade spectrum (13) carrying a constant flux of energy.

Finally, we would like to discuss the question of the spectrum structure in the pump region, in particular, for the case of the spectrally narrow pump. It was shown recently by Falkovich and Shafarenko[12] that, for wave systems with three-wave interaction, a spectrally narrow pump produces a sharp peak of the spectrum, even in the regime of a fully developed turbulence. Such a peak gives rise to a chain of peaks (the so-called pre-Kolmogorov asymptotics), which then turns into the power Kolmogorov spectrum. It was argued in Ref. 1 that for a four-wave interaction (as for the systems considered in this paper) such a chain of peaks should not exist. The question is whether a unique sharp peak could exist in a steady state under a spectrally narrow pump. In particular, such a peak was observed in the numerical simulations of two-dimensional optical turbulence (see Ref. 7). We found no sharp peak however narrow the pump was. This agrees with the previous numerical simulations presented in Ref. 8. It means that the steady values of $n(\omega)$ are so high that the typical time of nonlinear interaction $t_{nl}(\omega_0)$ is much less than the instability growth rate $\Gamma(\omega_0)$. So the spectrum value in the pump region could be estimated by continuing the Kolmogorov solutions there. It allows us to express the fluxes P and Q in terms of the amplitude and the position of the pump:

$$P = \int \Gamma(\omega)\omega^{3/2} n(\omega)\, d\omega \approx n_0 \omega_0^{3/2} \int \Gamma(\omega)\, d\omega \propto \Gamma^{2/3},$$

$$Q = \int \Gamma(\omega)\omega^{1/2} n(\omega)\, d\omega \approx n_0 \omega_0^{1/2} \Gamma \propto \Gamma^{2/3} \omega_0^{-1}.$$

Here Γ is the integral intensity of the pump.

IV. CONCLUSIONS

A steady spectra of weak Langmuir and optical turbulence are shown to be close to the Kolmogorov-like cascade spectra. The high-frequency part (13) carries a constant energy flux that has a value that is proportional to the pump growth rate in the power $\frac{3}{2}$ and is independent of pump frequency. The low-frequency part (10) carries a constant flux of wave action that has a value that is inversely proportional to the pump frequency.

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