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Bottleneck phenomenon in developed turbulence

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It is shown how viscosity increases turbulence level in the inertial interval by suppressing turbulent transfer.

If turbulence theory aspires to be in quantitative agreement with measurements, then it should be able to describe the behavior of correlation functions at finite Reynolds numbers. For an energy cascade in incompressible fluid, the energy spectrum can be derived from dimensional analysis as follows: $E(k)=\epsilon^{2/3}k^{-5/3}f(k/k_p)$. Here ϵ is the energy dissipation rate. The cutoff wave number k_p is proportional to the dissipation wave number $k_d=\epsilon^{1/4}\nu^{-3/4}$, which depends on the viscosity ν. Two asymptotics of the function $f(x)$ are more or less well defined: (i) $f(x)\to C$ as $x\to 0$, where C is the Kolmogorov constant;[1] (ii) $f(x)\to C'x^a e^{-x}$ as $x\to\infty$.[2]

One is tempted to presume that the function f decreases monotonically with k, which actually was used in most applications (except very recent ones; e.g. Ref. 3). That will be shown not to be true below: for a $3d$ energy cascade, the function f grows with k in the inertial interval, and only at $k\simeq k_p$ it starts to decrease. Some larger spectral density of energy compared to the Kolmogorov $\frac{5}{3}$ spectrum is due to the bottleneck phenomenon first discovered for acoustic turbulence:[4] A viscous suppression of small-scale modes removes some triads from nonlinear interaction. This makes nonlinear energy transfer less efficient, which leads to a pileup of energy in the inertial interval of scales. Here we describe this phenomenon and consider different examples of both wave turbulence and vortex turbulence in an incompressible fluid.

Formally, one can describe the bottleneck phenomenon as follows. Let us write the equation for the pair correlation function $F(k,t)$:

$$\frac{\partial F_k}{\partial t}=\int T(k,k_1)d\mathbf{k}_1-\nu k^2 F_k. \tag{1}$$

Here the transfer function $T(k,k_1)$ describes nonlinear interaction. We shall restrict ourselves to the case of the lowest nonlinearity when $T(k,k_1)$ is expressed via the triple correlation function. In the inviscid case, the equation has steady scale-invariant solution, with the pair correlator expressed by a power function $F(k)\propto k^{-s}$ for all k. If one substitutes that solution into (1), one finds that the viscous term grows with k faster than the nonlinear one so that one can introduce the dissipation wave number k_d that separates the inertial interval from the viscous interval of scales. All correlation functions drop sharply (exponentially) at $k>k_d$. If the contribution of the region $k_1\gg k$ in the transfer term is negative (as is the case for turbulent viscosity in a $3d$ energy cascade), then the absence of motions with $k_1>k_d$ gives a *positive* contribution onto the RHS of (1). It was first pointed out by Saffman that molecular viscosity suppress turbulent viscosity.[5] If that positive term prevails over the second negative term (that represents a direct effect of the viscosity) then the bottleneck phenomenon occurs: viscosity increases turbulence level in the inertial interval of scales.

If the interaction is local, then the contribution from given k_1 into $\partial F_k/\partial t$ decreases with the growth of the ratio k_1/k. For $k\ll k_d$, such a bottleneck correction to a Kolmogorov-like distribution is small, and can be found analytically by a simple perturbation theory, assuming the inviscid distribution is given. We shall show that those corrections are not exactly power functions: they have to contain a logarithmic factor, too.

As k approaches k_d, the bottleneck deviation from the inviscid distribution increases. Unfortunately, we cannot describe the form of the spectrum at $k\lesssim k_d$, where this deviation is not small and the perturbation theory fails.

Since we are interested here in small-scale behavior, then only a direct cascade is considered. From the Navier–Stokes equation one obtains the equation for the pair correlation function $\langle v_i(k)v_j(k')\rangle=F_{ij}(k)\delta(k+k')$,

$$\partial F_{ij}(k,t)/\partial tV=I_{ij}(k)-\nu k^2F_{ij}(k),$$

$$I_{ij}(k)=\mathrm{Re}\int\Gamma_{ilm}(k,k_1,k_2)J_{jlm}(k,k_1,k_2)$$

$$\times\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)d\mathbf{k}_1\,d\mathbf{k}_2. \tag{2}$$

Here J is the triple velocity correlator. The vertex is as follows: $\Gamma_{ilm}(k,k_1,k_2)=P_{ia}(k)P_{lb}(k_1)P_{mc}(k_2)\iota(k_b\delta_{ac}+k_c\delta_{ab})$, with the transverse projector $P_{ab}(k)=\delta_{ab}-k_ak_b/k^2$. A scale-invariant Kolmogorov solution $J_0(\lambda k,\lambda k_1,\lambda k_2)=\lambda^{-s}J_0(k,k_1,k_2)$ turns the transfer term I_k into zero for $s=7$.[1,7] It corresponds to $F_{0,ij}(k)$

$=(C/4\pi)\times P_{ij}(k)\epsilon^{2/3}k^{-11/3}$. We are looking for a steady solution ($\partial F/\partial t=0$) that is slightly deviated from the Kolmogorov one in the inertial interval $(k\ll k_p):J=J_0+\delta J$. Our first aim is to find δJ. Equation (2) gives

$$\int \Gamma_{ilm}(k,k_1,k_2)\delta J_{jlm}(k,k_1,k_2)\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)d\mathbf{k}_1\,d\mathbf{k}_2 + \delta I_{ij}(k,k_p)-\nu k^2F_{0ij}(k)=0. \tag{3}$$

Here $\delta I(k,k_p)=-\text{ Re }\int_{k_p}\Gamma(k,k_1,k_2)J_0(k,k_1,k_2)\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)dk_1\,dk_2$, where the integral is taken over the region $k_1>k_p$. It is shown below to describe the decrease of turbulent viscosity in the presence of molecular viscosity. Thus, we aim at finding the form and the value of the contribution into the transfer integral from the region $k_1\gg k$. It has been studied previously by using closures, i.e., expressing J via F in a more or less plausible way.[2] What is worth emphasizing here is that the form of the asymptotics can be established rigorously for any turbulent distribution:[8]

$$\lim_{k_1\gg k} J_{\alpha\beta\gamma}(k,k_1,k_1)\approx F_{\beta\mu}(k)k_\nu\Phi_{\alpha\mu\gamma\nu}(k_1). \tag{4}$$

Here $\Phi(k_1)$ can be expressed in terms of some series containing the pair correlator and the Green's function $G(k,\omega)$ in the ω representation. For example, the first term is $\Phi_0=F_{\alpha\mu}(k_1)G_{\gamma\nu}(k_1,0)$.[8] We designate $\Phi_{\alpha\mu\gamma\nu}(k_1)=P_{\alpha\mu}P_{\gamma\nu}\phi(k_1)$. For the Kolmogorov solution, $\phi(k_1)=C_1\epsilon^{1/3}k_1^{-13/3}$, with a yet unknown numerical factor C_1 of order unity. Substituting (4) into (3), we see that the small-scale contribution into the transfer function has the form $\nu_T(k_p)k^2F(k)$, i.e., it, indeed, corresponds to some turbulent viscosity:

$$\nu_T(k_p)=4\pi\int_{k_p}\phi(k_1)k_1^2\,dk_1=\nu C_1 3\pi\left(\frac{k_d}{k_p}\right)^{4/3}.$$

Thus we have a degeneracy: the nonlinear transfer gives the same k dependence as a molecular viscosity: $k^2F(k)$. This is due to the fact that $\Gamma\propto k$, and this is a peculiarity of incompressible fluid (as we shall see below, the k dependence of an ultraviolet contribution can be different for different turbulent systems). To make a conclusion about the sign of the effect of viscosity, it is not enough to calculate only the exponent of the ultraviolet contribution (as will be in considering wave turbulence), one should rather compare the numerical values of ν and $\nu_T(k_p)$. In our approach it is not worth calculating the numerical factor C_1 (that we cannot do anyway), since the contribution δI has been already written by neglecting factors of order unity while we put the correlation function to be zero at $k>k_p$, neglecting its actual form. The empirical fact that $k_p\ll k_d$ enables us to give a definite answer, nevertheless. According to various data, the ratio (k_p/k_d) is between 0.10 and 0.15.[3,9,10] Since $\nu_T(k_d)\simeq\nu$ by the very definition of k_d, then $\nu_T(k_p)>\nu$ at $k_p\ll k_d$. Thus, we see that the ultraviolet contribution prevails over the viscosity term. The same answer one gets estimating C_1 by the first diagram or using different versions of DIA or EDQNM:[10,11,2] $\nu_T(k_p)\approx 0.27\nu(k_d/k_p)^{4/3}$. Therefore, Eq. (3) has the form

$$\int \Gamma(k,k_1,k_2)\delta J(k,k_1,k_2)\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)d\mathbf{k}_1\,d\mathbf{k}_2 + C_2\nu k^{-5/3}\epsilon^{2/3}=0, \tag{5}$$

with some positive constant C_2. This is a scale-invariant integral equation for finding δJ in the inertial interval. The solution should be scale invariant as well. The scaling exponent of δJ can be readily established: $s=-17/3$. This means that $\delta J/J_0\propto(k/k_p)^{4/3}$. Now we should find $\delta F/F_0$. One can relate F and J by using the Wyld diagram technique or any closure (the answer below does not depend on it). Presuming that Kolmogorov values J_0 and F_0 satisfy that relation, one can linearize near the Kolmogorov solution and find nothing else but $\delta F/F_0\propto(k/k_p)^{4/3}$ if the divergences are absent. This simple statement is based on the fact that small corrections for different correlation functions should be linearly related. Substituting δF into (4,2) one gets, however, an ultraviolet logarithmic divergency in agreement with Refs. 8 and 12. It is a general rule that the distortion of the spectrum due to a remote region has a logarithmic factor (also see below). Finally, the viscous correction to the Kolmogorov spectrum in the inertial interval can be written as follows (cf. with Ref. 11):

$$\delta E(k)=E(k)(k/k_p)^{4/3}\ln^{-1}(k_p/k). \tag{6}$$

This formula is valid at $k\ll k_p$. As k approaches k_p, the correction grows slower. Above calculations have been based on the assumption that the unperturbed spectrum $E(k)$ is exactly the Kolmogorov $\frac{5}{3}$ spectrum. If one assumed some intermittency corrections, it does not qualitatively change our conclusions about bottleneck phenomenon. The form of the viscous correction depends on the spectrum: if the energy spectrum was $E(k)\propto k^{-s}$, then one can show that $\delta E(k)=E(k)(k/k_p)^{3-s}\ln^{-1}(k_p/k)$.

Spectrum flattening compared to the $\frac{5}{3}$ law has been previously observed both in the numerical simulations of the Euler equation[9] and its closures,[11,13,14] and in the experiments.[15] Note that the bottleneck phenomenon should be more pronounced for simulations using a superviscosity term that grows faster than k^2.

The same flattening should be observed for high-order correlation functions. Note that the bottleneck corrections grow with the order of the correlation functions: $\delta F^{(n)}/\delta F_0^{(n)}\propto n(k/k_p)^{4/3}$. For finite Reynolds number, it gives qualitatively the same type of anomalous scaling (deviations from a Kolmogorov scaling that grow with the number of the correlator) that is often assumed to be related to intermittency. This shows that an account of bottleneck corrections is necessary in interpreting the data of experiments and numerics in order to make correct statements about the inviscid limit.

Another possible application of the bottleneck phenomenon is in describing the extended scaling[16] that according to Ref. 17 takes place until the turbulence spectrum is less steeper than k^{-3}. Spectrum flattening due to a bottleneck should indeed extend the scaling region to smaller scales.

For the direct vorticity cascade in two dimensions, the ultraviolet contribution $T(k,k_1)$ is positive for $k_1\gg k$. This is the phenomenon of negative eddy viscosity discovered by Kraichnan.[18] The suppression of the modes with $k_1 > k_d$ thus gives a negative contribution as well as a usual viscosity term, so that the spectrum decays steeper than the Kolmogorov-like solution at any k.

Now we consider wave turbulence at low excitation level which can be described by the kinetic wave equation written in terms of the pair correlation function $\langle a(k,t)a^*(k',t)\rangle = n_k(t)\delta(k-k')$:[6] $\partial n_k/\partial t = I_k\{n_{k'}\} - \gamma_k n_k$. Here the functional I_k is a collision integral that describes nonlinear transfer.

We consider scale-invariant systems where $\omega_k \propto k^\alpha$. Kolmogorov-like spectrum of wave turbulence is the power distribution $n_k^0 = Dk^{-s_0}$ that turns the collision integral into zero: $I_k\{n_{k'}^0\}=0$. If the typical inverse time of nonlinear interaction $\tau^{-1}(k) = I_k/n_k \propto k^{-h}$ grows with k slower than γ_k, then the dissipation prevails at k larger than some k_d. We shall find the correction to the steady spectrum at $k \ll k_d$. We consider the case of a triple wave interaction when the angle-averaged collision integral is as follows:[6]

$$I(k) = \int_0^\infty (R_{k12} - R_{1k2} - R_{2k1})]dk_1\, dk_2,$$

$$R(k,k_1,k_2) = U(k,k_1,k_2)\delta(\omega_k - \omega_1 - \omega_2)\Theta(k-k_1) \times (n_1 n_2 - n_k n_1 - n_k n_2).$$

Here U is some positive function, its homogeneity index we designate β. A steady spectrum corresponds to a constant energy flux: $\int^q \omega_k k^{d-1} I(k)dk =$ const. The spectrum thus has the exponent $s_0 = (\beta + d + 2)/2$. We assume that the collision integral converges on power solutions $n_k \propto k^{-s}$ if the index s is a number out of the locality interval (s_1, s_2). If such an interval exists (local interaction) then $s_0 = (s_1 + s_2)/2$,[6] and the deviation of stationary distribution from the power law, induced by the effect of a remote sink, is small and may be found with the help of the perturbation theory that exploits the small parameter k/k_d.

The absence of waves at $k > k_d$ (we set $n_k \equiv 0$ at $k > k_d$) leads to a nonzero value of the collision integral at $k \ll k_d$:

$$\delta I_1(k) = 2D^2 \int_{k_d}^\infty U(k_1,k,k_0) k_0^{1-\alpha}\, dk_1 [(kk_0)^{-s_0} - (kk_1)^{-s_0} - (k_1 k_0)^{-s_0}] \propto D^2 k^{-\alpha-d}(k/k_d)^{s_0 - s_1}.$$

Here $k_0^\alpha = k_1^\alpha - k^\alpha$. We see that δI_1 is positive if $s_0 > \alpha$. This last condition is quite natural, since it means that the turbulence spectrum is steeper than an equilibrium Rayleigh–Jeans spectrum, so that the flux is directed downscale (direct energy cascade).[6] We should now compare δI_1 with $\gamma_k n_k^0 = \nu\, Dk^{2-s_0}$. The ratio of these two terms is proportional to $(k/k_d)^{s_0 - s_1 - 2 - h}$. The condition for δI_1 to dominate is thus

$$s_0 - s_1 - 2 - h < 0. \tag{7}$$

Physically, this means that the typical time of the interaction between waves with k and k_d is less than the time $\nu^{-1}k^{-2}$ of viscous dissipation.

As one can see, the ultraviolet contribution does not necessarily have the form of turbulent viscosity, as in an incompressible fluid. Since for a general turbulent system the dissipation term and the ultraviolet term have different k dependencies, then they differ by some power of a small factor k/k_d. As a result, to make a definite conclusion about the bottleneck existence, it is enough to know the scaling indices, and it is not necessary to calculate the prefactors, as in the preceding example.

The values of s_0, s_1, and h for various systems can be found in Ref. 6. Taking the most popular examples of wave turbulence under a three-wave interaction, one can see that the condition (7) for the bottleneck existence is valid for two- and three-dimensional sound and invalid for capillary waves. Further, we restrict ourselves with the former cases, where one can neglect the term $\gamma_k n_k$ in favor of δI_1 in the inertial interval. For the distribution n_k to be stationary, the addition δI_1, which owes its origin to the finite character of the inertial interval, must be compensated by a contribution δI_2 due to the small deviation of the solution from a pure power law $(n_k = n_k^0 + \delta n_k)$: $\delta I_2 = \hat{L}_k \delta n_k$, where $\hat{L}_k$ is the operator of the kinetic equation linearized with respect to δn_k. This integral operator is scale homogeneous: $\hat{L}_{\lambda k} = \lambda^{-h}\hat{L}_k$ with $h = \alpha + d - s_0$.

Thus, in order to determine δn_k, we should solve the linear integral inhomogeneous equation, $\delta I_1 + \delta I_2 = \delta I_1 + \hat{L}_k \delta n_k = 0$. Simple power counting gives the power function $\delta n_k \propto k^{-s_1}$, which is on the margin of uv divergency, so that a true solution has the following form:

$$\delta n_k = C'' Dk^{-s_0}(k/k_d)^{s_0 - s_1} \ln^{-1}(k_d/k). \tag{8}$$

Here C'' is a dimensionless constant, that can be calculated in any particular case, sign $C'' = \text{sign}(s_0 - \alpha)$. Therefore, for the spectra with positive energy flux $(s_0 > \alpha)$, the δn_k value grows with k, i.e., the distribution has a somewhat smoother slope. Of course, at $k \simeq k_d$, a sharp falloff of n_k should take place, which now cannot be described in terms of the perturbation theory. The dependence of $\log n_k$ on $\log k$ has an inflection point.

To conclude, it is shown that the bottleneck is a quite general physical phenomenon connected with the nature of energy exchange between modes.

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