Intermittent distribution of heavy particles in a turbulent flow

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Abstract

The retardation of weakly inertial particles depends on the acceleration of the ambient fluid. The concentration n of the particles is determined by the divergence of Lagrangian acceleration which we study by a direct numerical simulation. We show that the second moment of the concentration coarse-grained over the scale r behaves as an approximate power law : $\langle \overline{n}_r^2 \rangle \sim r^\alpha$. We study the dependencies of the exponent α on the Reynolds number, of the Stokes number and of the settling velocity. We find that the theoretical lower bound previously suggested roughly captures the order of magnitude and the dependencies on the parameters. Our numerics give the value two to four times larger than the theoretical lower bound. We show that the discrepancy grows as the Reynolds number increases and analyze the possible physical mechanism behind it.

Inertial particles suspended in a non-uniform flow are distributed non-uniformly even when flow is incompressible. Air bubbles are concentrated in vortex cores while heavy particles are expelled from vortices by a centrifugal force. The phenomenon is used, for instance, for flow visualization [2]. In a turbulent flow, the distribution of particles must be characterized statistically [1]. Particularly important is the second moment of the concentration (or two-particle probability distribution) since it determines, for instance, the collision rate of water droplets in clouds [3, 4]. For a non-uniform distribution, the second moment can be much larger than the mean concentration squared. Our main goal in this paper is to find the enhancement factor $\langle n^2\rangle/\langle n\rangle^2$.

Particles of density ρ_0 and radius a in a fluid of density ρ and of viscosity ν have their velocity $\mathbf{v}$ related to the fluid velocity $\mathbf{u}$ by the equation ($\rho_0 >> \rho$) [5]:

$$d\mathbf{v}/dt = (\mathbf{u} - \mathbf{v})/\tau_s + \mathbf{g} \ , \tag{1}$$

where $\tau_s = (2a^2\rho_0)/(9\rho\nu)$. We consider a dilute (passive) case when particles do not modify the fluid velocity. We presume that τ_s is short compared to the Kolmogorov time, $\tau_K = \langle\omega^2\rangle^{-1/2}$ which is the smallest time scale associated with turbulence. The Stokes number, defined as $St \equiv \tau_s/\tau_K$ is restricted here to small values: $St \lesssim 0.3$. In this case, the particle velocity is well approximated by $\mathbf{v} = \mathbf{u} - \tau_s d\mathbf{u}/dt + \tau_s\mathbf{g}$ where $d\mathbf{u}/dt \equiv \partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}$ is the Lagrangian acceleration. The velocity field $\mathbf{v}$ is *compressible* even when $\mathrm{div}\,\mathbf{u} = 0$ [5] : $\mathrm{div}\,\mathbf{v} = -\tau_s\nabla\cdot(\mathbf{u}\cdot\nabla)\mathbf{u} = -\tau_s tr(m^2)$, where $m_{ij} = (\partial u_j/\partial x_i)$. Since $tr(m^2) = s^2 - \omega^2/2$, where s^2 is the strain and ω is the vorticity, then vorticity (strain) dominated regions tend to repel (attract) particles.

Gravity can be characterized by the dimensionless ratio of the velocity difference at the Kolmogorov scale, $(\mathcal{E}\nu)^{1/4}$ to the settling velocity $\mathbf{v}_g = \tau_s\mathbf{g}$: $\epsilon \equiv (\mathcal{E}\nu)^{1/4}/v_g$ (small-scale Froude number). Here $\mathcal{E}$ is the energy dissipation. Note that the product of $\epsilon \cdot St \equiv \epsilon_0$ is independent of the particle size a : $\epsilon_0 = \mathcal{E}^{3/4}/(g\nu^{1/4})$. We restrict ourselves here to cases where turbulence dominates compared to gravity ($0.01 \lesssim \epsilon_0$).

The continuity equation in the Lagrangian frame, $dn/dt = \partial_t n + \mathbf{v}\cdot\nabla n = -n\,\mathrm{div}\,\mathbf{v}$, gives $n(\mathbf{x},t) = n(0,0)\exp\tau_s\int_0^t tr(m^2)(t')dt'$ where the integral is along the Lagrangian trajectory. Such continuous description with Brownian diffusion neglected is valid only for the concentration field $\overline{n}_r$ coarse-grained over some scale r exceeding both a and the diffusion scale [1, 4]. When the flow of the ambient fluid is turbulent, the velocity gradients (and the matrix m) are correlated on the scale comparable to the viscous scale η. It is thus straightforward to show that expansions and contractions of a small volume can accumulate only until the largest size is stretched up to η [1, 4]. In other words, one can show that concentration is uniform on the scales far exceeding η because there the turbulent diffusion wins over compressibility [6], the conclusion also supported by observations [7]. If we denote t_r the time needed to stretch fluid element from r to η then [1, 4]

$$\overline{n}_r(\mathbf{x},0) = n_0 \exp\Big(\tau_s \int_{-t_r}^{0} tr(m^2)(t')dt'\Big) \tag{2}$$

The statistical weight associated with each trajectory is $1/n$ so the space-averaged second moment is as follows

$$\langle \overline{n}_r^2 \rangle = \left\langle n_0^2 \exp\Big(\tau_s \int_{-t_r}^{0} tr(m^2)(t')dt'\Big) \right\rangle \ , \tag{3}$$

where the (Lagrangian) average is over the set of trajectories. Since the stretching is exponential at the scales less than η then $t_r \propto \ln(\eta/r)$ so that $\langle \overline{n}_r^2 \rangle$ is expected to be a power law: $(\eta/r)^\alpha$ [1, 4]. The main problem is to find the exponent α as a function of the dimensionless parameters St, ϵ, Re. Physically, α must be determined by the ratio of the backward-in-time rate of volume expansion to the rate of line stretching in the $\mathbf{v}$-flow [1, 4]. Without gravity and for small St, the line stretching rate is close to the most negative Lyapunov exponent of the fluid flow which we denote λ_3. Gravity decreases the line stretching rate by decorrelating velocity gradients in the reference frame of falling particles. It was suggested in [4] to describe the line stretching rate of the $\mathbf{v}$-flow by the formula $\lambda_3 \min\{1, \epsilon\} \equiv \lambda_3 \tilde{\epsilon}$ which is exact in both limits of weak and strong gravity. At $St\tilde{\epsilon} \ll 1$, it was argued in [4] that the volume expansion rate can be evaluated via the variance of $tr(m^2)$ in the reference frame of the moving particle so that $\alpha_{th} = St^2 \int \langle tr(m^2)(0) tr(m^2)(t')\rangle dt'/2\tilde{\epsilon}\lambda_3$. Since only the single-time probability density function (pdf) $\mathcal{P}(s)$ of a single velocity derivative, $s = \partial u/\partial x$, is usually available experimentally, it is of great practical importance to estimate α in terms of $\mathcal{P}(s)$. It was suggested in [4] to replace the Lagrangian integral of the fourth-order moment by the single-time third moment which gives the crude estimate (without gravity): $\alpha_{est} \simeq \langle |\nabla \mathbf{u}|^3 \rangle \tau_s^2/\lambda_3 = 15^{3/2}\tau_s^2 \int \mathbf{P}(s) s^3\, ds/\lambda_3$ integrated over $s < \tau_s^{-1}$. The account of gravity modifies this estimate as follows: $\alpha_{est} \simeq 15^{3/2}\tau_s^2 \int \mathbf{P}(|\sigma|)\sigma^4 \tau_c\, d\sigma/\tilde{\epsilon}\lambda_3$. Here $\sigma < \tau_s^{-1}$, $\tau_c = \tau_s + (\sigma + \sqrt{\sigma/\tau_K}\epsilon^{-1})^{-1}$ and $\mathbf{P}(|\sigma|) = (1+\sigma^2/s_*^2)\mathcal{P}(|s| = |\sigma| + \sigma^2/s_*)$ with $s_* = \tau_s^{-1}\min\{1, \epsilon^2/St\}$ — see [4] for the details of derivation. Another theoretical prediction was suggested in [8] based on the phenomenological model: $\alpha_{ph} = 4St^2/5$. In this note, we calculate (3) by direct numerical simulations, demonstrate that $\overline{n}_r \propto r^{-\alpha}$ and compare the resulting value of the exponent α with $\alpha_{th}, \alpha_{est}, \alpha_{ph}$.

Since we consider $St \ll 1$, then $|\mathbf{v} - \mathbf{u} - \mathbf{v}_g| \sim St|\mathbf{u} + \mathbf{v}_g| << |\mathbf{u} + \mathbf{v}_g|$. Based on this fact, we follow the trajectory with the velocity $\mathbf{u} + \mathbf{v}_g$ instead of $\mathbf{v}$ which overestimates the time heavy particles spend around vortices, and underestimates the time they spend in straining regions. This is likely to underestimate the growth of inhomogeneities so our results are expected to give an estimate from below for α.

Simulating a high-Re flow, while resolving the concentration of inertial particles over a large range of scales *below* the Kolmogorov length scale is in practice difficult. The method used here allows us to determine quantities such as $\overline{n}_r$ without such resolution requirements. Numerically, an Eulerian velocity field is generated by standard direct numerical simulations (DNS) methods using a pseudo spectral code. The flow is periodic in all three dimensions in space, and maintained in a statistically steady state by applying a forcing at the lowest wavenumbers. A full description of the code can be found in [9]. The number of grid points was varied in the range $32^3 - 256^3$, allowing us to vary the Reynolds number in the range $20 \leq R_\lambda \leq 130$, while maintaining an adequate resolution (the highest wavenumber k_{max} represented in the simulation is kept in the range $1.4 \leq k_{max}\eta \leq 2$).

At a given time, a set of N_l lagrangian trajectories is released. Lagrangian trajectories are followed using the algorithm developed in [10]. The compression between time $-t_r$ and 0 of a fluid parcel of size η to a small scale Along with the locations and velocities of the particles, we follow the transformation of (infinitesimal) line elements, $\delta\updownarrow(t)$. This can be conveniently done by introducing the (dimensionless) tensor W, defined by : $\delta\updownarrow_i(t) = W_{ij}(t)\delta\updownarrow_j(0)$, satisfying : $dW_{ij}/dt = m_{ik}W_{kj}$, as well as its inverse : $dW^{-1}_{ij}/dt = -W^{-1}_{ik}m_{kj}$ [11]. The compression between time $-t_r$ and 0 of a fluid parcel of size η to a small scale r implies that the norm of W^{-1} grows by a factor (η/r) during this lapse of time. By monitoring the tensor W^{-1}, we are therefore able to determine the final location $\mathbf{x}$ of a trajectory, starting at $\mathbf{x}_0$, such that the compression of a fluid element between the initial $t = 0$ and the final time $t = t_r$ is equal to a given value (η/r). This allows us to determine the quantities X_r and $\overline{n}_r$ without having to integrate the equation of motion for the fluid particles backwards in time.

At $\epsilon > 1$ and $St \leq 0.32$, the integration time $T \approx 60\tau_K$ was enough to let $\gtrsim 95\%$ of all trajectories to reach a ratio $\eta/r = 300$ and that time was found sufficient so that the expression (3) saturates and thus gives an accurate estimate of $\langle \overline{n}_r^2 \rangle$. In fact, using a smaller integration time, ($T \approx 45\tau_K$) in several runs, in particular at our highest resolution 256^3, didn't introduce significant biases in the results. At lower ϵ, the time necessary to obtain a large enough stretching increased. The runs presented in this work were long enough, so that $\gtrsim 90\%$ of all trajectories reach a ratio $\eta/r = 300$. To improve the statistics, the same procedure was run several times. Overall, a minimum of 5.10^5 trajectories were followed, and the flow was sampled for a minimum of $\gtrsim 6$ eddy turnover times.

The stretching factor $r_c = \ln(|W^{-1}\delta\mathbf{l}_0|/|\delta\mathbf{l}_0|)$ and the Lyapunov exponent $\lambda_3 = -\lim_{t\to\infty} r_c/t$ were calculated first. Without gravity, for all R_λ in our range we have found $-\lambda_3 \approx 1/6\tau_K$ in a good agreement with [11]. With gravity, the stretching rate is a monotonically increasing function of R_λ. At fixed R_λ, the stretching rate is observed to decrease sharply when ϵ decreases (not shown). This can be understood since a fast settling velocity makes the stretching experienced along particles' trajectories less correlated, thus diminishing the stretching. At small ϵ, we have found $\lambda_3 \propto \epsilon$.

Figure 1 presents the dependence of the second moment of the coarse-grained density (3) as a function of the scale ratio η/r. The larger the value of the Stokes number the stronger is the growth of $\langle \overline{n}_r^2 \rangle$ as a function of η/r. The curves are approximately straight in our log-log coordinates supporting thus the prediction of the power law made in [1] and [4]. The exponent $\alpha = \frac{d\ln\langle \overline{n}_r^2 \rangle}{d\ln(\eta/r)}$ is shown on Figure 2 as a function of Stokes number, for several values of the Reynolds number R_λ ($\epsilon_0 = \infty$). Note that the approximate power law for the pair correlation function of the concentration has been observed before for larger scales and larger St [12, 13], our data are complementary as they include gravity, allow to track the power law below η and also correspond to smaller St. Numerically, our values of the exponent are between those of [12] and [13].

Our results show a strong growth of α when R_λ increases. The exponent is observed to diminish strongly when the settling velocity grows. Interestingly, we observe that for $\epsilon_0 \lesssim 0.02$, the exponent ceases growing when the Reynolds number increases. The actual α (and the level of concentration non-uniformities) is well-estimated by α_{th} of [4] at least at not very high R_λ. For example, without gravity $\alpha_{th}/\alpha = 1.04, 1.45, 1.82, 1.69$ for $R_\lambda = 21, 47, 83, 105$ respectively. Let us stress that α depends on the fluid velocity in a nonlinear and nonlocal way even at small St. It is remarkable that the formula based on the lagrangian variance of velocity derivatives gives a reasonable estimate as predicted in [4]. Note that the phenomenological estimate $\alpha_{ph} = 4St^2/5$ gives at least ten times smaller values, which tells that the simple model of [8] (which uses only the second-order single-time velocity statistics) does not describe well the concentration below the viscous scale. The estimate α_{est} (which uses the whole single-time velocity pdf [4]) works better as it underestimates α by a factor between 2 and 4 as shown at Fig.3. Those results support the prediction of [4] that α_{est} gives a lower bound for α. We see that the discrepancy grows with R_λ but decreases with the increase of settling velocity. Without gravity, α_{est} captures well the dependence on St so that one may suggest to correct the underestimation of R_λ-dependence by a semi-empirical factor depending only on R_λ: $\alpha = \alpha_{est} c(R_\lambda)$. While our data suggest that c grows with R_λ a bit faster than linear, further extensive study is needed here. Particularly interesting is to understand the physical mechanism of that R_λ-dependence. We believe that the single-time single-component estimate α_{est} underestimates the concentration fluctuations of inertial particles because it does not take into account the peculiarities of the statistics of the divergence of the Lagrangian acceleration, $tr(m^2)$. We present the pdf of the (dimensionless) volume compression rate, $X_r = \tau_K \int^{t_r} tr(m^2)dt$ for $\eta/r = 2^p$, p varying between 1 (lower curve) to 8 (upper curve) for the case without gravity. We see that the pdf at small scales is very non-Gaussian and asymmetric. When $r \lesssim \eta$, the distribution of X_r reduces to the distribution of $tr(m^2) = s^2 - \omega^2/2$. Due to the presence of very intense vortex tubes, this distribution is very skewed towards negative values. The skewness remains negative and large throughout the range of scales covered in this study. This effect can only be explained by a long correlation time along lagrangian trajectories. The settling velocity plays an important role in this process, by decorrelating the straining field. As the coherence time becomes very small ($\epsilon \ll 1$), the distribution $P(X_r)$ becomes substantially narrower and more symmetric.

The subtle properties of small scale turbulence have a direct and very remarkable impact on the phenomenon of preferential concentration. The spectacular increase of the exponent as a function of the Reynolds number, which is for our purpose the most interesting property, results from the intermittency of the turbulent velocity field. The dependence of the exponent α as a function of the Reynolds R_λ leads in turn to important consequences in cloud physics. One of the most intriguing problems of cloud physics is the so-called condensation-collision bottleneck (see e.g.[4]): condensation relatively fast produces droplets with sizes in tens of microns while collision rate (based on $\langle n\rangle^2$) is too small to explain the observed rain initiation time (sometimes less than an hour). Droplet sizes $a \simeq 10 \div 40$ microns typically correspond to small Stokes numbers, $St \simeq 0.02 \div 0.3$. Our results show that the enhancement factor $\langle n^2 \rangle / \langle n \rangle^2 = (\eta/r)^\alpha$ can exceed order of magnitude already for such small St. Note that the enhancement factor multiplies the Saffman-Turner collision rate due to turbulence [14]. Our findings thus support the idea that turbulence may indeed significantly enhance collisions and influence the rain initiation time in turbulent warm clouds.

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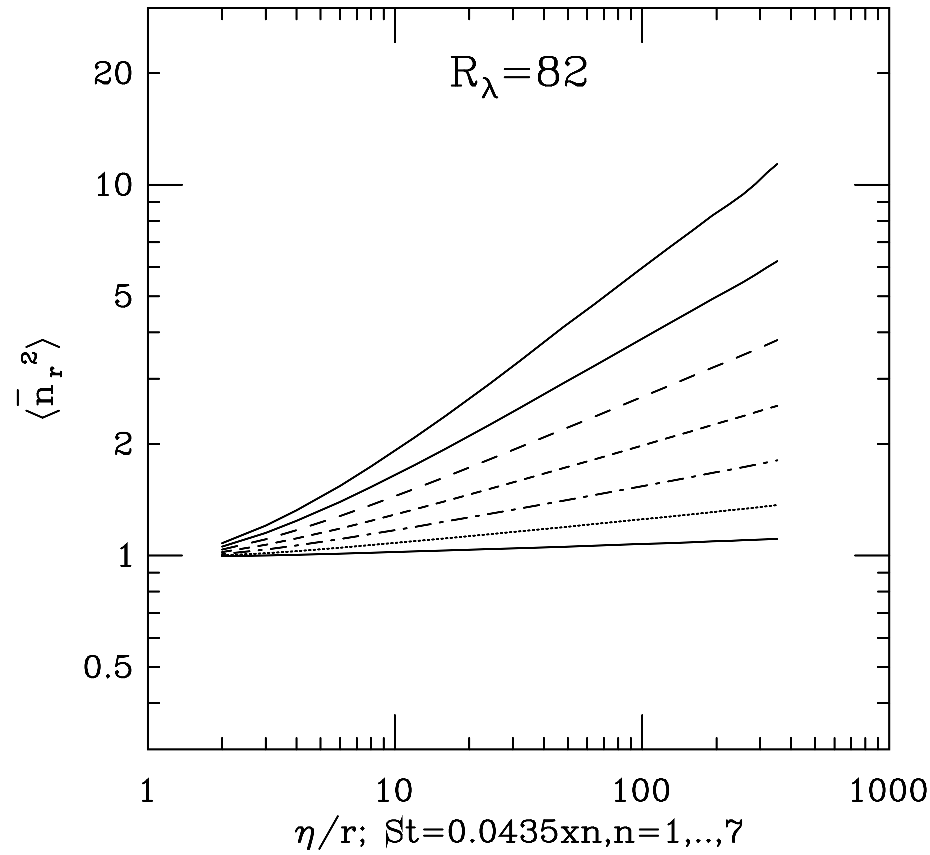

FIG. 1: The variance of the concentration fluctuation $\langle \overline{n}_r^2 \rangle$ as a function of r, at $R_\lambda = 82$ and for $St = p \times 0.435$.

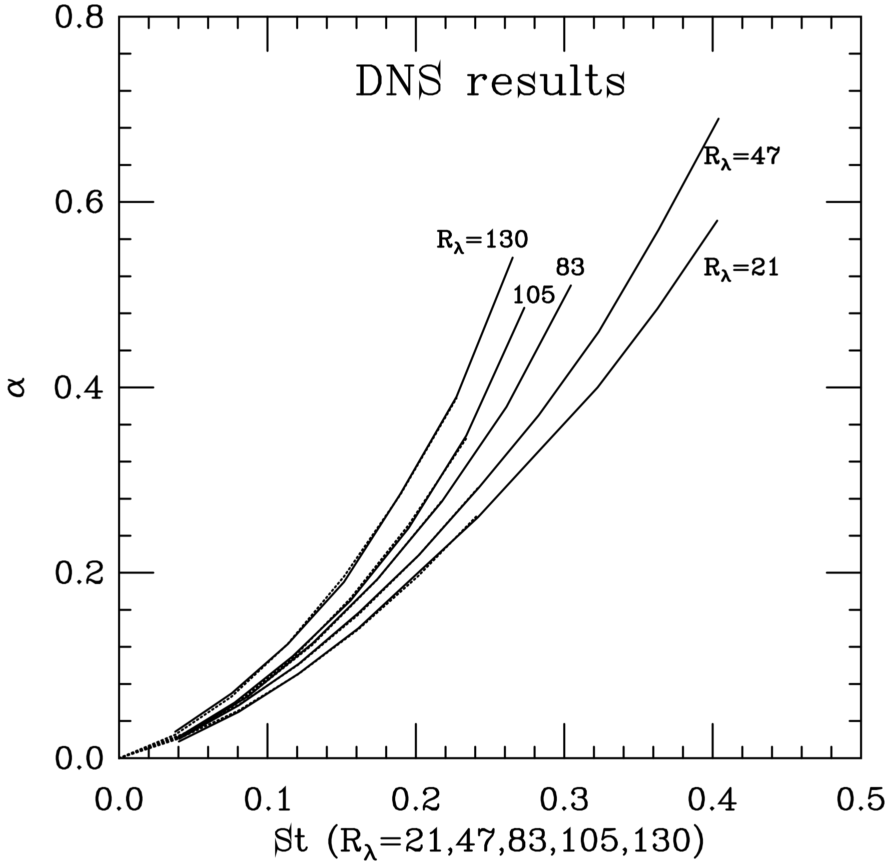

FIG. 2: The exponent α as a function of the Reynolds number, R_λ and the stokes number, St

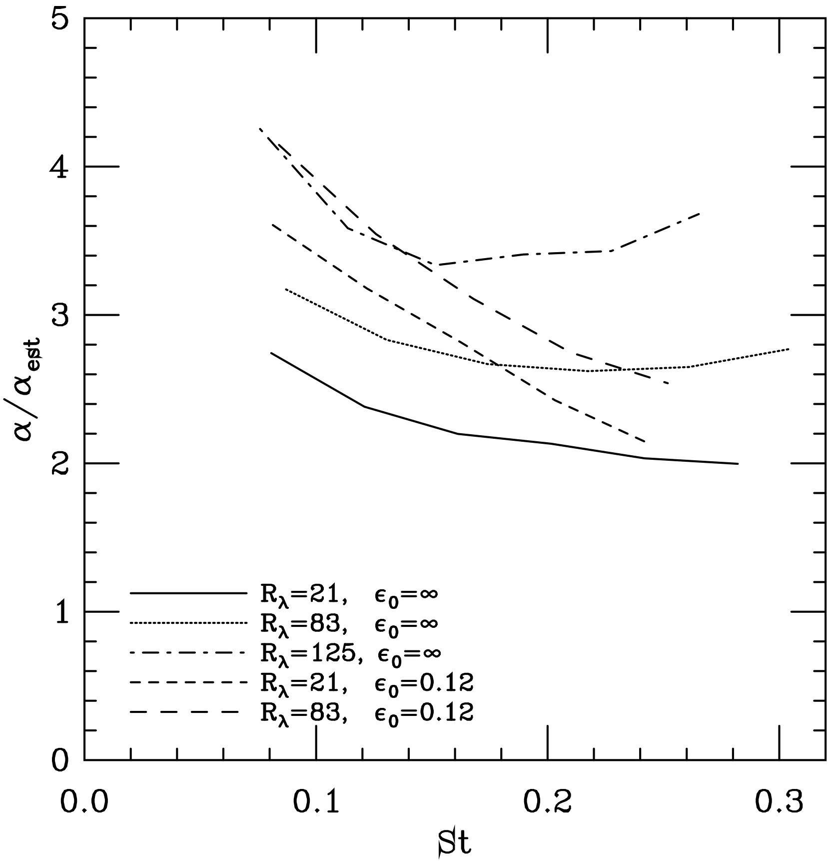

FIG. 3: Comparison between the exponent α and the predicted value α_{est}, see text.

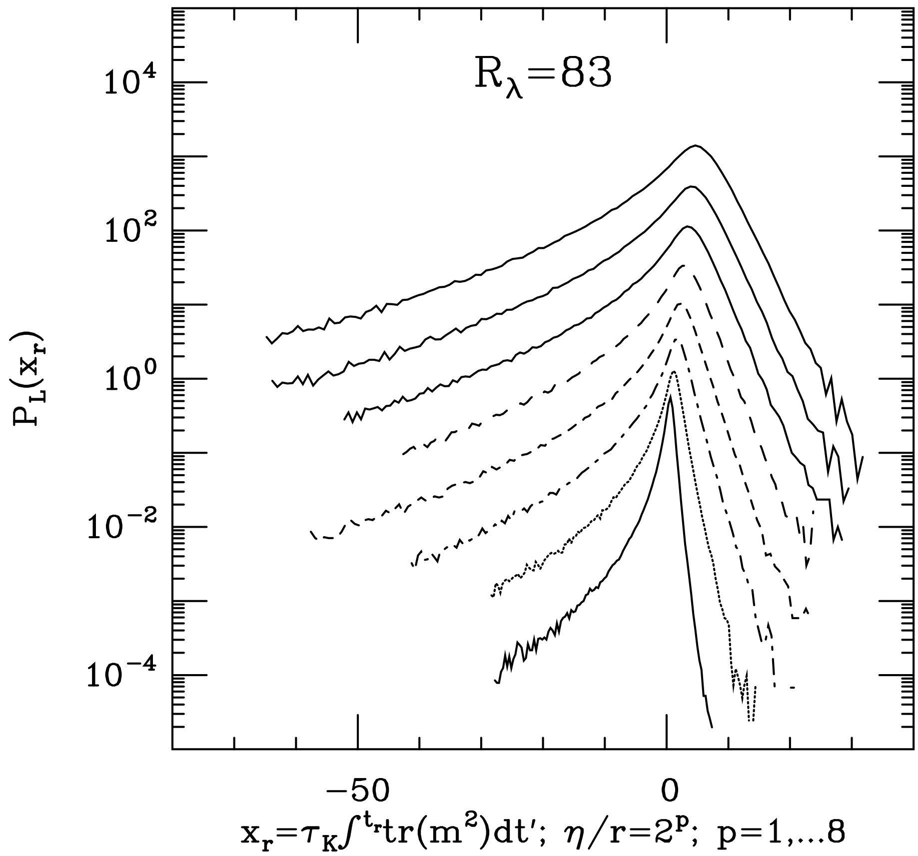

FIG. 4: Pdf of the quantity $X_r = \int_0^{t_r} tr(m^2)(t')dt'$ for several values of r, at $R_\lambda = 83$ and for $\epsilon_0 = \infty$.