# 12. Cartan matrix and Dynkin diagrams

A very useful construct in dealing with semi-simple Lie algebras is the *Cartan matrix* $\mathsf{A}$, defined by $\mathsf{A}_{ij} = 2\alpha^{(i)} \cdot \alpha^{(j)} / \alpha^{(j)} \cdot \alpha^{(j)}$, where $\{\alpha^{(i)}\}$ are the simple roots of the algebra. For a rank $\ell$ algebra, the Cartan matrix is an $\ell \times \ell$ matrix, with all diagonal elements equal to 2 and all off-diagonal elements zero or negative. It is generally not symmetric. (The numerator is symmetric in $i$ and $j$, but the denominator involves only $j$.)

From the previous discussion of the "magic formula", it follows that the non-diagonal elements of the Cartan matrix are limited to $\mathsf{A}_{ij} = 0, -1, -2, -3$. From the Schwarz inequality $(\alpha^{(i)} \cdot \alpha^{(j)})^2 \leq (\alpha^{(i)} \cdot \alpha^{(i)})(\alpha^{(j)} \cdot \alpha^{(j)})$, and from the linear independence of the simple roots (so that $\alpha^{(i)}$ cannot be proportional to $\alpha^{(j)}$ for $i \neq j$), it may be concluded that the product of off-diagonal elements placed symmetrically about the main diagonal is limited by $\mathsf{A}_{ij}\mathsf{A}_{ji} < 4$.

With the aid of the Cartan matrix, all the roots can be expressed in terms of the simple roots. It is sufficient to determine the positive roots, from which the negative roots can be obtained by an overall sign change. Every positive root can be written, as proved before, in the form $\sum_i k_i \alpha^{(i)}$, where $\{\alpha^{(i)}\}$ are the simple roots and the $\{k_i\}$ are non-negative integers. The quantity $k = \sum_i k_i$ is called the *level* of the root. The simple roots belong to the first level, $k = 1$, with one of the $k_i$ being unity and the rest zero. The other positive roots can be built up by adding simple roots consecutively. This will be referred to as the *building-up process* for roots.

For any element $\gamma$ of root space, its *Dynkin indices* are defined as $\Delta_i = 2\gamma \cdot \alpha^{(i)} / \alpha^{(i)} \cdot \alpha^{(i)}$, where the $\{\alpha^{(i)}\}$ are the simple roots of the algebra. The Dynkin indices of a root are integers and the positive root $\sum_j k_j \alpha^{(j)}$ has Dynkin indices $\Delta_i = \sum_j k_j \mathsf{A}_{ji}$, in terms of elements of the Cartan matrix. (Note that the rows of the Cartan matrix are the Dynkin indices of the simple roots.) Each time a simple root $\alpha^{(l)}$ is added to the root $\sum_j k_j \alpha^{(j)}$, the integer $k_l$ increases by 1 and the $l^{\text{th}}$ row of the Cartan matrix is added to the row of Dynkin indices to generate the Dynkin indices of the new root.

According to the "magic formula", the Dynkin indices of a root are equal to $m-p$, where $m$ and $p$ define the ends of a simple-root string containing that root. If $p > 0$ for some simple root, a new root can be produced by adding that simple root to the existing root. If $p = 0$ for some simple root, a new root can not be generated in this way. But $\Delta_i = m_i - p_i \Longrightarrow p_i = m_i - \Delta_i$, where the index $i$ indicates which simple root is being tested for addition to the existing root. Since the roots are being built up by systematically adding simple roots, the value of $m_i$ for the simple root $\alpha^{(i)}$ is known at each step, by counting backwards.

> [Note that at level $k = 1$, where all the roots are simple, roots can be generated by subtracting the simple root $\alpha^{(i)}$ from itself twice (since both 0 and $-\alpha^{(i)}$ are roots) but no roots can be generated by subtracting a simple root from a different simple root (since the difference of simple roots is not a root). So the row of $m$ values for the simple root $\alpha^{(i)}$ at level 1 is $\{0, 0, \ldots, 0, 2, 0, \ldots, 0, 0\}$, where the sole non-zero entry is a 2 in the $i^{\text{th}}$ position.]

The procedure can be summarised as follows. At each level, list the roots already found, together with their Dynkin indices and $m$ values. (Note that each root has a row of $\ell$ Dynkin indices and a row of $\ell$ values of $m$, one for each simple root.) Compute for each root the values of $p_i = m_i - \Delta_i$. For each $p_i > 0$, add the corresponding simple root to the existing root to get a new root and add the corresponding row of the Cartan matrix to the existing Dynkin indices to get the Dynkin indices of the new root so generated. When all new roots at that level (one level above the starting level) have been found, repeat the process at this new level. The building-up process terminates when all $p_i = 0$ for all roots at a given level.

> [As an **illustrative example**, consider the Lie algebra $G_2$, which is a rank-2 algebra with the Cartan matrix $\mathsf{A} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$. At level $k = 1$, the simple roots are represented by $\Delta = (2, -3)$ for $\alpha^{(1)}$ and $\Delta = (-1, 2)$ for $\alpha^{(2)}$. From $\alpha^{(1)}$, admissible roots are obtained by subtracting $\alpha^{(1)}$, to get the root 0, or $2\alpha^{(1)}$, to get the root $-\alpha^{(1)}$, but no admissible root is obtained by subtracting $\alpha^{(2)}$, since the difference of two simple roots is not a root. So $m = (2, 0)$ for $\alpha^{(1)}$. Similarly, $m = (0, 2)$ for $\alpha^{(2)}$. Taking the difference, $p = (0, 3)$ for $\alpha^{(1)}$ and $p = (1, 0)$ for $\alpha^{(2)}$. Since $p_2 > 0$ for $\alpha^{(1)}$, while $p_1 > 0$ for $\alpha^{(2)}$, level $k = 2$ roots can be obtained by adding $\alpha^{(2)}$ to $\alpha^{(1)}$ or by adding $\alpha^{(1)}$ to $\alpha^{(2)}$. Both steps produce the same positive root, $\alpha^{(1)} + \alpha^{(2)}$, with $\Delta = (1, -1)$, obtained by adding the first row of $\mathsf{A}$ to $\Delta$ for $\alpha^{(2)}$ or the second row of $\mathsf{A}$ to $\Delta$ for $\alpha^{(1)}$. Admissible roots can be obtained from this sole level 2 root by subtracting $\alpha^{(1)}$ or $\alpha^{(2)}$ once each, so it has $m = (1, 1)$. Subtracting $\Delta$ produces $p = (0, 2)$, so a further positive root can be found by adding $\alpha^{(2)}$, since $p_2 > 0$. At level $k = 3$ there is thus the root $\alpha^{(1)} + 2\alpha^{(2)}$, with $\Delta = (0, 1)$, obtained by adding

the second row of $\mathsf{A}$ to the previous $\Delta$. Admissible roots can be obtained from this one by subtracting $\alpha^{(2)}$ twice, so $m = (0, 2)$ and $p = (0, 1)$. It is possible to add $\alpha^{(2)}$ yet again, since $p_2 > 0$, so level $k = 4$ has the root $\alpha^{(1)} + 3\alpha^{(2)}$, with $\Delta = (-1, 3)$, again adding the second row of $\mathsf{A}$ to the previous $\Delta$. Admissible roots are found by subtracting $\alpha^{(2)}$ three times, so now $m = (0, 3)$, leading to $p = (1, 0)$, allowing the addition of $\alpha^{(1)}$, since now $p_1 > 0$. This produces the level $k = 5$ root $2\alpha^{(1)}+3\alpha^{(2)}$, with $\Delta = (1, 0)$ and $m = (1, 0)$, so that $p = (0, 0)$ and the process terminates. The algebra $G_2$ has six positive roots, six corresponding negative roots and two zero roots, so is of dimension 14. Note that, at each stage of the process of building up the positive roots, the entries in $p$ indicate how much further it is possible to progress along the corresponding $\alpha^{(i)}$-strings.]

This example can be structured as follows.

$$\mathsf{A} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

$$\begin{array}{lcccc}
\underline{k=1} & \begin{array}{c}\alpha^{(1)}\\ m=(2,0)\\ \Delta=(2,-3)\\ p=(0,3)\end{array} & & \begin{array}{c}\alpha^{(2)}\\ m=(0,2)\\ \Delta=(-1,2)\\ p=(1,0)\end{array} & \\
 & \searrow(2) & & (1)\swarrow & \\
\underline{k=2} & & \begin{array}{c}\alpha^{(1)}+\alpha^{(2)}\\ m=(1,1)\\ \Delta=(1,-1)\\ p=(0,2)\end{array} & & \\
 & & \searrow(2) & & \\
\underline{k=3} & & & \begin{array}{c}\alpha^{(1)}+2\alpha^{(2)}\\ m=(0,2)\\ \Delta=(0,1)\\ p=(0,1)\end{array} & \\
 & & & \searrow(2) & \\
\underline{k=4} & & & & \begin{array}{c}\alpha^{(1)}+3\alpha^{(2)}\\ m=(0,3)\\ \Delta=(-1,3)\\ p=(1,0)\end{array} \\
 & & & & (1)\swarrow \\
\underline{k=5} & & & \begin{array}{c}2\alpha^{(1)}+3\alpha^{(2)}\\ m=(1,0)\\ \Delta=(1,0)\\ \underline{p=(0,0)}\end{array} &
\end{array}$$

The Cartan matrix also determines the Lie products (commutation relations) of the algebra. For each simple root $\alpha^{(i)}$, the auxiliary quantities $e_i = E_{\alpha^{(i)}}$, $f_i = 2E_{-\alpha^{(i)}}/(\alpha^{(i)} \cdot \alpha^{(i)})g_{\alpha^{(i)},-\alpha^{(i)}}$ and $h_i = 2H_{\alpha^{(i)}}/(\alpha^{(i)} \cdot \alpha^{(i)})$ are defined. It is easily checked that these quantities obey the commutation relations $[e_i, f_j] = h_i\delta_{ij}$ (where the factor $\delta_{ij}$ arises because the difference $\alpha^{(i)} - \alpha^{(j)}$ of simple roots is not a root), $[h_i, e_j] = \mathsf{A}_{ji}e_j$ (using the result $[H_\alpha, E_\beta] = (\alpha \cdot \beta)E_\beta$) and $[h_i, f_j] = -\mathsf{A}_{ji}f_j$.

Since all positive roots arise as combinations of the simple roots (again determined by the Cartan matrix), the corresponding generators $e$ and $f$ can be defined as commutators (like $[e_i, e_j]$, for instance) or multiple commutators (like $[[e_i, e_j], e_j]$, for instance) of the simple generators, and appropriate $h$'s can be defined by analogy with the above. Then the Cartan matrix again fixes the further commutation relations, like $[h_i, [e_j, e_k]] = (\mathsf{A}_{ji} + \mathsf{A}_{ki})[e_j, e_k]$, where the Jacobi identity has been used. (Note that the same non-simple root may be obtained by different multiple commutators. In this case, some specific choice of the representative multiple commutator must be made.)

Negative roots will have generators determined by the parallel multiple commutators of the negatives of the simple roots. In this sense, the algebra is fixed by its Cartan matrix.

A very helpful pictorial representation of the Cartan matrix is supplied by *Dynkin diagrams*. In such a diagram, each simple root is represented by a small circle, the shorter roots by a filled circle, and the circles are numbered. Circles numbered $i$ and $j$ are connected by $\mathsf{A}_{ij}\mathsf{A}_{ji}$ straight lines. Using the known limitations on the Cartan matrix elements $\mathsf{A}_{ij}$, the Cartan matrix can be reconstructed from the Dynkin diagram.

Simple Lie algebras have connected Dynkin diagrams, while semi-simple algebras have disconnected Dynkin diagrams, one piece for each of the simple ideals whose direct sum is the semi-simple algebra. There exist only a limited number of possible connected Dynkin diagrams, hence only a limited number of simple Lie algebras.

> [As an **illustrative example**, consider the algebra $G_2$ discussed above, with the Cartan matrix $\mathsf{A} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$. It has two simple roots, of different lengths. (The elements $\mathsf{A}_{12} \neq \mathsf{A}_{21}$ differ only in their denominators, which are respectively the squared length of simple root number 2 and the squared length of simple root number 1. Since $\mathsf{A}_{12}$ is larger, simple root number 2 is shorter.) The first is represented by an open circle, the second by a closed circle. The two circles are connected by three straight lines. The Dynkin diagram is
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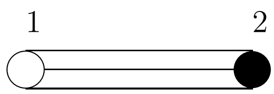
> 
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In order to establish the set of permitted Dynkin diagrams, a set of vectors $\{\gamma_i\} \in \mathcal{H}_0^*$ is considered. The matrix of integers $\mathsf{M}_{ij} = 2\gamma_i \cdot \gamma_j / \gamma_j \cdot \gamma_j$ is defined and represented by a diagram of numbered circles, with circles $i$ and $j$ connected by $\mathsf{M}_{ij}\mathsf{M}_{ji}$ straight lines. The set of vectors is defined as *allowable* if it satisfies the three conditions: (i) the $\{\gamma_i\}$ are linearly independent, i.e. the determinant of $\mathsf{M}$ is non-zero; (ii) $\mathsf{M}_{ij} \leq 0$ for $i \neq j$; (iii) $\mathsf{M}_{ij}\mathsf{M}_{ji} = 0, 1, 2, 3$. The following sequence of results is then established.

1. Any subset of an allowable set is allowable.

   [This follows immediately from the definition of an allowable set.]

2. An allowable set has more circles than joined pairs.

   [Define $\gamma = \sum_i \gamma_i/\sqrt{\gamma_i \cdot \gamma_i}$. Since the $\{\gamma_i\}$ are linearly independent, $\gamma \neq 0$, so $\gamma \cdot \gamma > 0$, which can be written

$$\begin{aligned} 0 &< \sum_i (\gamma_i/\sqrt{\gamma_i \cdot \gamma_i}) \cdot (\gamma_i/\sqrt{\gamma_i \cdot \gamma_i}) \\ &\quad +2\sum_{i<j} \gamma_i \cdot \gamma_j/\sqrt{(\gamma_i \cdot \gamma_i)(\gamma_j \cdot \gamma_j)} \\ &= \text{number of circles} - \sum_{i<j} \sqrt{\mathsf{M}_{ij}\mathsf{M}_{ji}} \Longrightarrow \\ \text{number of circles} &> \sum_{i<j} \sqrt{\mathsf{M}_{ij}\mathsf{M}_{ji}} \\ &> \text{number of joined pairs,} \end{aligned}$$

   since $\mathsf{M}_{ij}\mathsf{M}_{ji} \geq 1$ for a joined pair. (Note the sign change of the off-diagonal sum, since $\gamma_i \cdot \gamma_j < 0$ but $\mathsf{M}_{ij}\mathsf{M}_{ji} > 0$.)]

3. The diagram of an allowable set has no closed loops.

   [The loop alone would constitute a subset of the allowable set. But a loop has a number of circles less than or equal to the number of joined pairs, so is not allowable, by result 2. This would contradict result 1.]

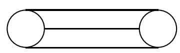
4. Suppose the diagram of an allowable set contains a chain of single links. Shrinking this chain to a single circle produces an allowable set.

[Let the circles in the chain represent $\beta_1, \beta_2, \ldots, \beta_n$ and define $\beta = \sum_i \beta_i$. Then $\beta \cdot \beta = \sum_i \beta_i \cdot \beta_i + 2\sum_{i<j} \beta_i \cdot \beta_j$ $= n(\beta_i \cdot \beta_i) - (n-1)(\beta_i \cdot \beta_i) = \beta_i \cdot \beta_i$, using the fact that single links imply vectors of equal length and $\mathsf{M}_{ij} = -1 \Longrightarrow 2\beta_i \cdot \beta_j = -\beta_j \cdot \beta_j$. If the chain is linked to the circle $\gamma$ at one end, then $\gamma \cdot \beta = \sum_i \gamma \cdot \beta_i = \gamma \cdot \beta_1$ or $\gamma \cdot \beta_n$. If the set containing the $\{\beta_i\}$ was allowable, then the set with $\beta$ replacing all the $\{\beta_i\}$ is equally allowable.]

5. For an allowable set, no more than three lines are attached to any circle.

   [Suppose the circle $\gamma_0$ is joined to $\gamma_1, \gamma_2, \ldots, \gamma_n$. Then $\gamma_i \cdot \gamma_j = 0$ for $i, j \neq 0$, since there are no closed loops, i.e. the $\{\gamma_i, i \neq 0\}$ are mutually orthogonal. Since $\gamma_0$ is linearly independent of the $\{\gamma_i\}$, its squared length is strictly greater than the sum of the squares of its projections along the mutually orthogonal $\gamma_i$'s, $\gamma_0 \cdot \gamma_0 > \sum_i (\gamma_0 \cdot \gamma_i / \sqrt{\gamma_i \cdot \gamma_i})^2$, i.e. $\sum_i \mathsf{M}_{0i}\mathsf{M}_{i0} < 4$.]

6. The only allowable triple line occurs in the two-circle diagram



   where the three lines join the two circles.

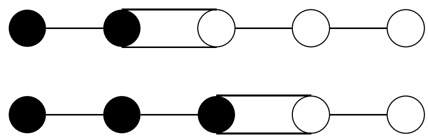
   [This follows directly from the previous result.]

7. The diagram of an allowable set contains at most one circle connected with three single lines and at most one double-line segment, but not both.

   [If this rule were violated, it would be possible, by shrinking a chain to a point, to produce a vertex with more than three lines.]

   It follows from the last two results that there can be no more than two different lengths of vectors in an allowable set. From now on, the shorter vectors will be indicated by filled circles.

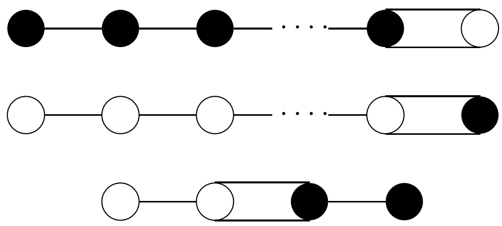
8. The matrices corresponding to the diagrams



have vanishing determinants and are not allowable.

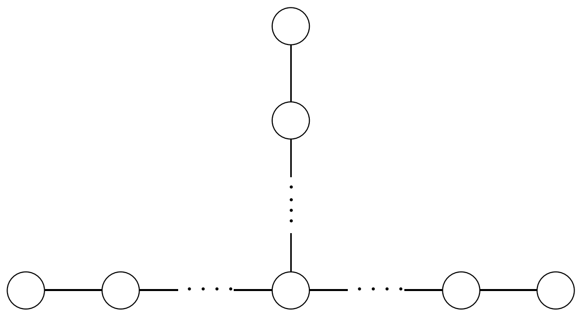
[ The matrix of the first diagram is $\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -2 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$.
Summing the first and fifth columns, plus twice the second and fourth columns, plus three times the third column produces a column of zeroes, hence a vanishing determinant.
The second diagram has the matrix $\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -2 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$.
Summing the first and fifth rows, twice the second and fourth rows and three times the third row produces a row of zeroes, hence a vanishing determinant.]

9. The only allowable diagrams containing a double line are



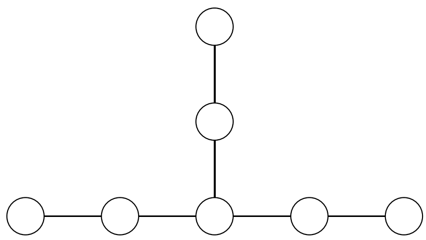
[Any other diagram with a double line contains one of the non-allowed diagrams of the previous step as a subset.]

10. The only allowed branched diagrams are



[The diagram can contain only one three-line vertex, no double or triple lines and no loops.]

11. The diagram

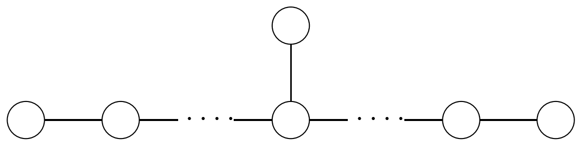


has vanishing determinant and is not allowable.

[The associated matrix is $\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$.

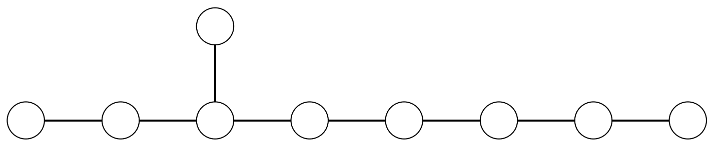
Summing the first, fifth and seventh columns, plus twice the second, fourth and sixth columns, plus three times the third column produces a column of zeroes, hence a vanishing determinant.]

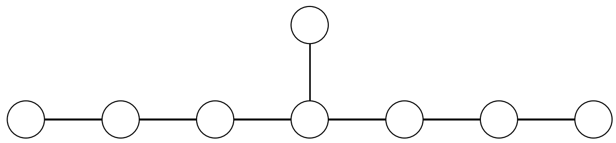
12. The only allowable branched diagrams remaining are



[Any other branched diagram contains the previous non-allowable diagram as a subset.]

13. Finally, the diagrams



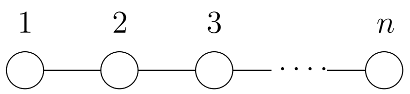


have vanishing determinants and are not allowable.

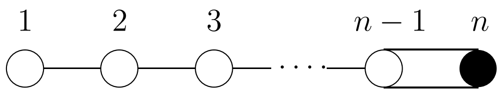
[Confirmed by direct evaluation of the determinants.]

The final classification of all allowable diagrams is:

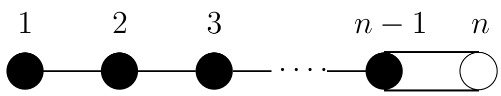
$\mathcal{A}_n$:



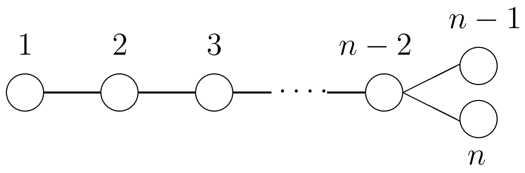

$\mathcal{B}_n$:



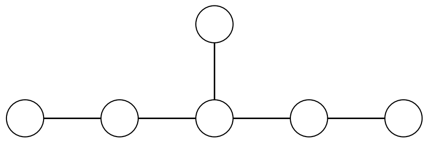

$\mathcal{C}_n$:



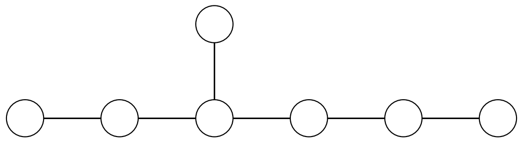
$\mathcal{D}_n$:



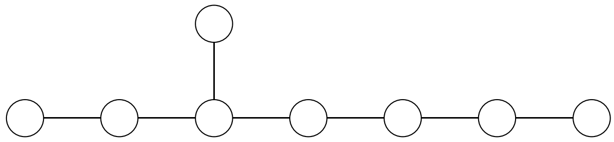

$\mathcal{E}_6$:



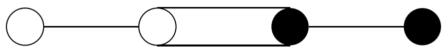
$\mathcal{E}_7$:



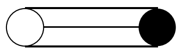
$\mathcal{E}_8$:



$\mathcal{F}_4$:



$\mathcal{G}_2$:



The above are all the allowable Dynkin diagrams which could be associated with simple Lie algebras. In fact, each one of these diagrams does have a corresponding Lie algebra. The families of algebras denoted $\mathcal{A}_n, \mathcal{B}_n, \mathcal{C}_n, \mathcal{D}_n$ are known as the *classical Lie algebras*, while the remaining five algebras, $\mathcal{E}_6, \mathcal{E}_7, \mathcal{E}_8, \mathcal{F}_4, \mathcal{G}_2$, are known as the *exceptional Lie algebras*. In every case, the subscript in the designation of the Lie algebra is the rank of the algebra.

The classical Lie algebras denoted by appropriate Dynkin diagrams derive from the complex extensions of certain matrix algebras.

- The Dynkin diagram of $\mathcal{A}_n$ is the Dynkin diagram of $\mathsf{su}(n+1)$ (in fact, of $\mathsf{su}(n+1-m, m)$ for any $m \leq n+1$);
- the Dynkin diagram of $\mathcal{B}_n$ is that of $\mathsf{so}(2n+1)$ (in fact, of $\mathsf{so}(2n+1-m, m)$ for any $m \leq 2n+1$);
- the Dynkin diagram of $\mathcal{C}_n$ is that of $\mathsf{sp}(2n)$; and
- the Dynkin diagram of $\mathcal{D}_n$ is that of $\mathsf{so}(2n)$ (in fact, of $\mathsf{so}(2n-m, m)$ for any $m \leq 2n$).

Certain isomorphisms can be read off from the identity of corresponding Dynkin diagrams.

> [$\mathcal{B}_2$ and $\mathcal{C}_2$ have the same Dynkin diagram, and $\mathsf{so}(5) \cong \mathsf{sp}(4)$; $\mathcal{A}_3$ and $\mathcal{D}_3$ have the same Dynkin diagram, and $\mathsf{su}(4) \cong \mathsf{so}(6)$; similarly, $\mathsf{su}(2) \cong \mathsf{so}(3) \cong \mathsf{sp}(2)$. Note that the Dynkin diagram for $\mathcal{D}_2$
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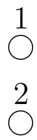
> 
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> is not connected, so $\mathcal{D}_2$ is semi-simple, not simple. It is a direct sum $\mathcal{B}_1 \oplus \mathcal{B}_1$ and $\mathsf{so}(4) \cong \mathsf{so}(3) \oplus \mathsf{so}(3)$.]