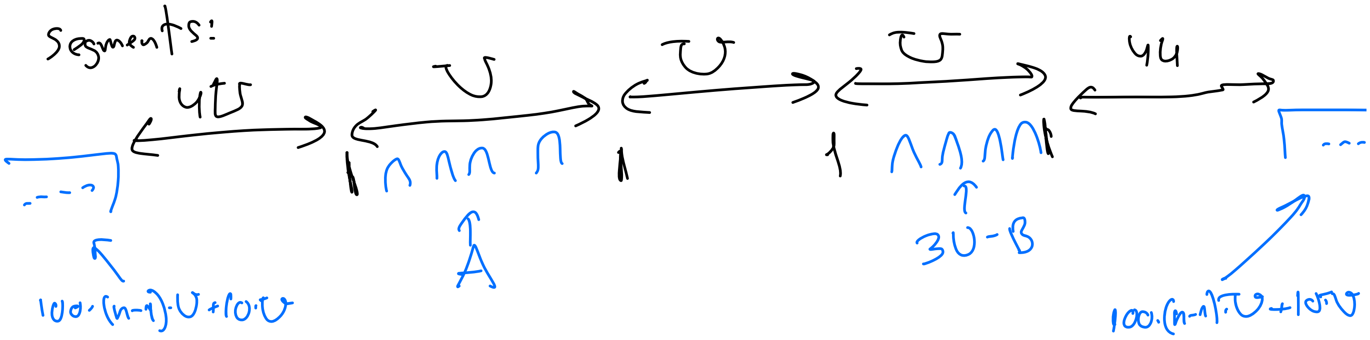
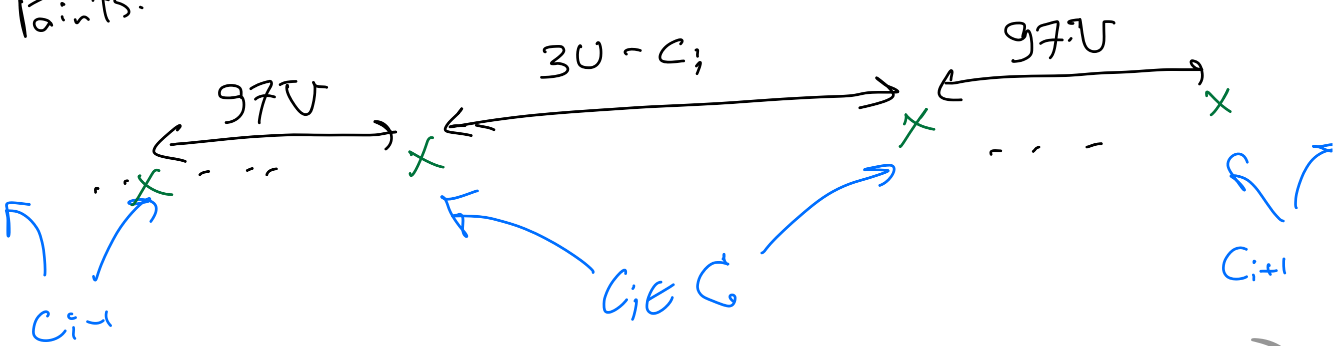
(See slides for 3SUM → Geometric Problems.)

Col-3SUM' → Segments Containing Points

Segments:



Points:

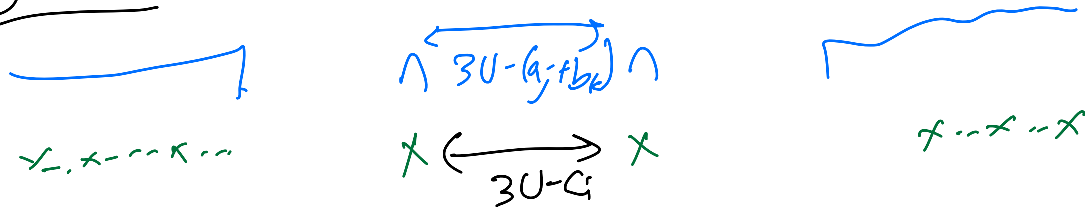


$\forall i: \leq (n-1)\cdot 100\cdot U$

but:

$> 100\cdot(n-1)\cdot U + 17\cdot U$

Claim 1: 3SUM $\Longrightarrow$ covering



Claim 2: covering $\Longrightarrow$ 3SUM

- long segments cannot cover all points
- some pair $x \xleftrightarrow{3U-c_i} x$ is covered by $A \cup B \ldots$
- must have $a_j + b_k = c_i$.

Next: 3SUM in $O(n + U \lg U)$ time

then: 3SUM $\longrightarrow$ Conv-3SUM.

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* FFT solves in $O(U \lg U)$:

Polynomial Multiplication | Convolution

Input: $p(x)=\sum_{i=0}^{U-1} a_i x^i$

$q(x) = \sum b_i x^i$

Output:

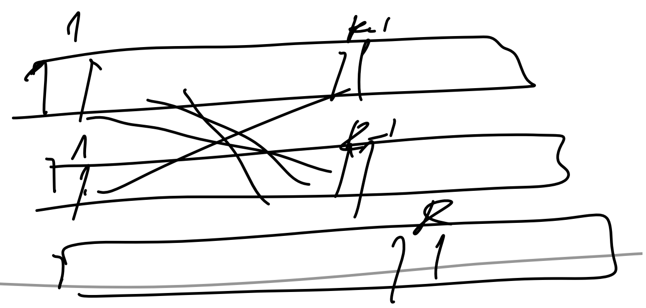
$$(p\cdot q)(x) = \sum_j^{2U-1} \Big(\sum_i a_i b_{j-i}\Big) x^j$$

Input: arrays of len U

A, B

Output: $C = A \odot B$ s.t.

$\forall j \in [2U]: \; C[j] = \sum_{i=1}^{U} A[i]\,B[j-i]$



— $O(n + U\log U)$ alg for Positive-3SUM':

- let $A[i] = B[i] = \begin{cases} 1 & i \in S \\ 0 & i \notin S \end{cases}$
- Compute $C = A \odot B$
- "yes" iff $\exists j$ s.t. $j \in S$ and $C[j] \geq 1$.

Correctness: $C[j] \geq 1 \iff \exists i : A[i] \geq 1,\; B[j-i] \geq 1$

$\iff \exists i : i \in S,\; j-i \in S.$

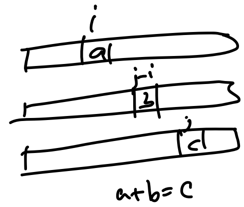
3SUM': given $S \subseteq [1, U = n^{3.1}]$

are there $a, b, c \in S$ s.t. $a+b=c$?

Convolution-3SUM: given arrays of length n A, B, C with numbers in $[1, U]$.

are there $i, j \in [n]$ s.t.

$A[i] + B[j-i] = C[j]$?



We saw (in HW): Conv-3SUM $\to$ 3SUM

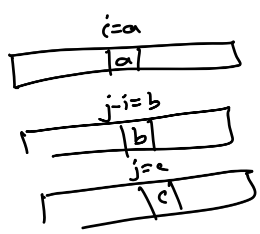
3SUM in $T(n)$ time $\Rightarrow$ Conv-3SUM in $O(T(n))$ time

Thm (Patrascu STOC'10): 3SUM $\to$ Conv-3SUM

Proof (simplified and optimized by [Chan-He SOSA'20])

" Conv-3SUM in $T(n)$ time $\Rightarrow$ 3SUM in $T(n\log^3 n) \log n$ time "

1st idea:



put a in $A[a]$...

problem: array is too long...
n^3 not $\sim n$.

2nd idea: hash $x \to h(x)$
and put a in $A[h(a)]$

want:

1. $h(x) = O(n \log^3 n)$

2. $a+b=c \Rightarrow h(a) + h(b) = h(c)$

3. few collisions

why? let $S_y = \{x \in S : h(x) = y\}$

what if $S_{h(a)} = \{a, a', a'', \ldots\}$ is large?

3rd idea: choose one at random:

$$A[i] = \begin{cases} \text{random } x \text{ in } S_i \\ +\infty \text{ if } S_i = \emptyset \end{cases}$$

- $B[i], C[i]$ defined similarly.

take: $h(x) = (x \bmod p)$ for $p \in_R [10 n \lg^3 n, 20 n \lg^3 n]$

1. ✓
2. almost: $(a \bmod p) + (b \bmod p) = (a+b) \bmod p$ or $(a+b) \bmod p + p$.
3. Lemma: $a \neq a' \Rightarrow \Pr[a = a' \bmod p] \leq \frac{\lg U}{10 n \lg^3 n / \lg(n \lg n)} \leq \frac{1}{10n}$

Reduction:
- repeat $10^7 \lg n$ times:
  - pick random $p \in [10 n \lg^3 n, 20 n \lg^3 n]$

  define $h(x) = x \bmod p$.
  - set $A[i] = \begin{cases} \text{random } x \text{ in } S_i \\ +\infty \text{ if } S_i = \emptyset \end{cases}$

$B[i], C[i]$ same way

- if Conv-3SUM(A,B,C) = yes
  return yes.

(⊛ also try with C' s.t. $C'[i+p] = C[i]$.)

<u>time:</u> ✓

<u>Correctness:</u>

<u>def.</u> - a is "good" if $|S_{h(a)}| \le 10$
- a is "bad" if $|S_{h(a)}| > 10$
- a is "chosen" if $A[h(a)] = a$

similarly: b is chosen if $B[h(b)] = b$...
(same for c)

<u>Claim:</u> if $a+b=c$ and a,b,c are chosen
then Conv-3SUM(A,B,C) = yes.

<u>proof:</u> Clear ✓

what is $\Pr[a,b,c \text{ are chosen}]$?

$\Pr[a \text{ is bad}] \le \frac{1}{10}$ [Markov's]

$$\Pr[\text{a is chosen} \mid \text{a is good}] \geq \frac{1}{10}$$

$$\Pr[\text{a is chosen}] \geq \frac{9}{100}$$

$$\Pr[\text{a, b, c are chosen}] \geq \left(\frac{9}{100}\right)^3 \geq \frac{1}{10^6}.$$