Def: $A \xrightarrow[ab]{} B$

A fine-grained (a,b)-reduction from A to B is an alg $\mathcal{A}^B$ for A with oracle acess to B, s.t.:

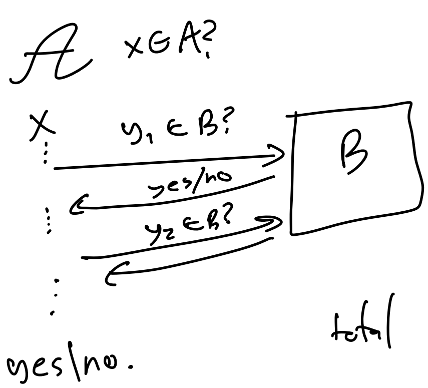
$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall$ input x of size n:

1. $\mathcal{A}^B(x)$ is correct. w.p. $\geq 1 - 1/n^{10}$.
2. $\mathcal{A}^B(x)$ takes $O(n^{a-\delta})$ time.
3. Let $y_1, \dots, y_t$ be the oracle calls, $|y_i| = n_i$,
   then
$$\sum_{i=1}^{t} n_i^{b-\varepsilon} = O(n^{a-\delta})$$

Thm: If $A \xrightarrow[ab]{} B$ and B is in $O(n^{b-\varepsilon})$ time
then A is in $O(n^{a-\delta})$ time.

thm: $A \xrightarrow[ab]{} B$ and $B \xrightarrow[bc]{} C$ then $A \xrightarrow[ac]{} C$

(Turing-Reduction)



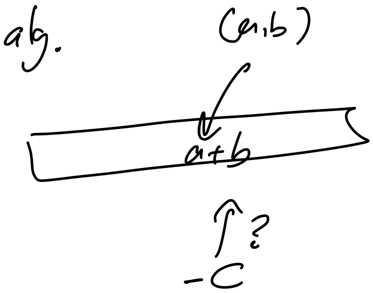
total time: $O(n^{a-\delta}) + \sum_{i=1}^{t} |y_i|^{b-\varepsilon}$

---

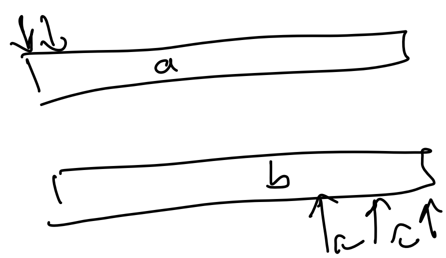
integers

3SUM: given a set of n integers $S \subseteq [-U, +U]$, is there $a, b, c \in S$ s.t. $a+b+c=0$?

- $O(n^2)$ algs. **Conj:** no $O(n^{2-\epsilon})$ alg.



k-SUM: given $S \subseteq [-U, +U]$ are there $x_1, \dots, x_k \in S$ s.t. $\sum_{i=1}^{k} x_i = 0$?



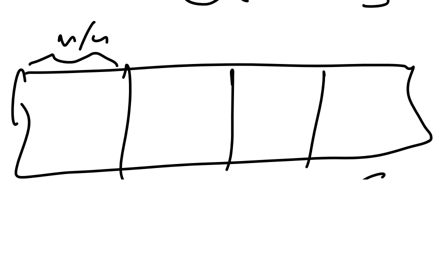
- $O(n^{\lceil k/2 \rceil})$
- $n^{o(k)}$ for k-SUM
  $\Rightarrow$ SAT is in $2^{o(n)}$ time (refutes ETH)

3SUM-Finding: given S ... return (a,b,c) s.t. $a+b+c=0$, if exists.

[Recall: "Finding $\to$ Decision" in NP]

Thm: If 3SUM is in $T(n)$ time, then 3SUM-Finding is in $O(T(n) \lg n)$ time.

pf:



$S_1 \quad S_2 \quad S_3 \quad S_4$

- let $S_i' = S \setminus S_i$. note: $|S_i'| = \frac{3}{4} \cdot n$
- Call 3SUM(S_i') $\forall i \in [4]$.
- Recurse on first S_i' that is yes, or return fail.
- if $|S| = O(1)$, solve in $|S|^3$ time.

* Correctness: easy. ✓
* time: $T_{Find}(n) \le 4 \cdot T_{Dec}(n) + T_{Find}\left(\frac{3}{4} n\right)$

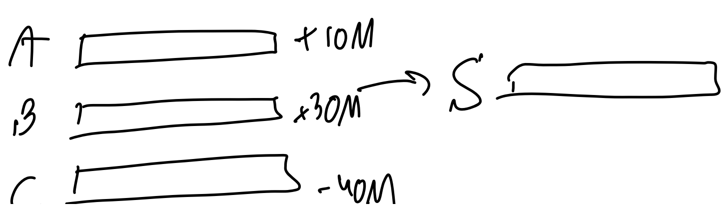
easy: $O(T_{Dec}(n) \cdot \log n)$

* (almost) never worry about Finding vs. Decision!

$$\text{3SUM} \underset{\text{recursing}}{\overset{\text{by def}}{\longleftrightarrow}} \text{3SUM-Finding}$$

Colorful-3SUM: given 3 sets of n numbers $A, B, C \subseteq [-U, +U]$ is there $a \in A, b \in B, c \in C$ s.t. $a + b + c = 0$?

Thm 1: 3SUM in $T(n)$ time $\Rightarrow$ Colorful 3SUM in $O(T(n))$.



$M = \max(A \cup B \cup C)$

$$
\begin{aligned}
S = &\{a + 10\cdot M \mid a \in A\} \\
&\cup \{b + 30\cdot M \mid b \in B\} \\
&\cup \{c - 40\cdot M \mid c \in C\}
\end{aligned}
$$

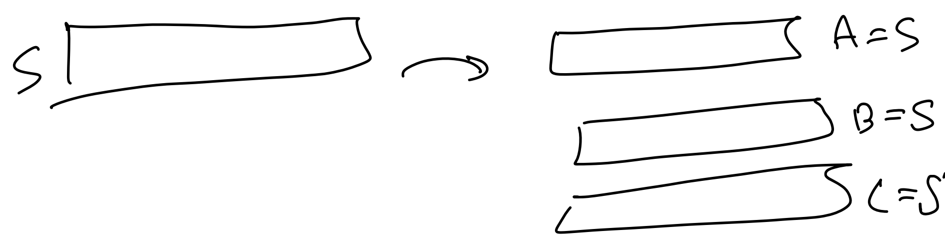
<u>Time:</u> ✓

<u>Correctness</u>: sum of any 3 with reps in $\{10, 30, -40\}$ is >10 away from 0.

$$
\begin{tikzcd}
\text{Colorful-3SUM} \arrow[r, "\text{add } M"] & \text{3SUM} \arrow[l, bend left, "?"]
\end{tikzcd}
$$

<u>Thm 2:</u> If Col-3SUM in $T(n)$ time, then 3SUM is in $\tilde{O}(T(n))$ time.

<u>pf:</u>



only proves 3SUM-with-Duplicates

$\downarrow$

Col-3SUM.

<u>trick:</u> Color-Coding (Alon-Yuster-Zwick '94)

<u>reduction:</u>

... times:

- repeat $100 \lg n$ ...
  - $\forall x \in S$ let $col(x) = \begin{cases} 1 & \text{w.p. } 1/3 \\ 2 & \text{w.p. } 1/3 \\ 3 & \text{w.p. } 1/3 \end{cases}$
  - let $A = \{x \in S \mid col(x) = 1\}$

    $B = \{x \in S \mid col(x) = 2\}$

    $C = \{x \in S \mid col(x) = 3\}$
  - if Col-3SUM(A,B,C) = yes, return yes
- return no.

<u>time:</u> ✓

<u>correctness</u>: $\forall$ any a,b,c: $\Pr\left[\begin{matrix} col(a)=1 \\ col(b)=2 \\ col(c)=3 \end{matrix}\right] = \left(\frac{1}{3}\right)^3$

$$\Rightarrow \Pr[\text{never}] \le \left(1 - \left(\frac{1}{3}\right)^3\right)^{100 \lg n}$$

$$\le \left(\frac{1}{2}\right)^{5 \lg n} \le \frac{1}{n^5}.$$

"no" $\Rightarrow$ "no"

"yes" $\Rightarrow$ "yes" w.p. $\ge 1 - \frac{1}{n^5}$.

$$\begin{tikzcd} \text{3SUM} \arrow[r, bend left, "\text{by def}"] \arrow[d, "\text{Color coding}"'] & \text{3SUM-Finding} \arrow[l, bend left, "\text{recursing}"] \\ \text{Col-3SUM} \arrow[u, bend right, "\text{add } M"'] & \end{tikzcd}$$

add ...

3SUM-with-Duplicates

Copy 3 times

k-path n^k vs $2^k \cdot n$

3SUM': given S are there $a,b,c \in S$ s.t. $a+b=c$?

Col-3SUM': given A,B,C … $a \in A$, $b \in B$, $c \in C$ s.t. $a+b=c$?

(α,β,γ)-3SUM: given S are there $a,b,c \in S$ s.t. $\alpha \cdot a + \beta \cdot b + \gamma \cdot c = 0$?

Col-(α,β,γ)-3SUM: given A,B,C … … ?

3SUM $\equiv$ $(1,1,1)$-3SUM

3SUM' $\equiv$ $(1,1,-1)$-3SUM

(α,β,γ)-3SUM

3SUM'

?

3SUM

Color Coding

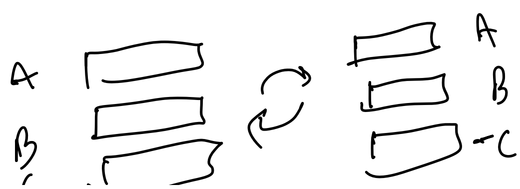
?

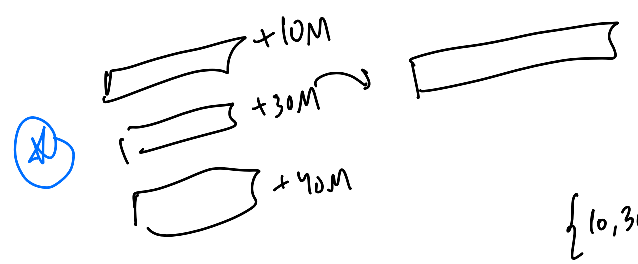
Col Cod

Col-$(\alpha\beta\gamma)$-3SUM

Col-3SUM'

Col-3SUM





$\{10, 30, 40\}$
cannot pick 3 with duplicates
s.t. $x+y=z$.

works easily unless $\alpha+\beta+\gamma=0$.
hard example $a+b=2c$ $(1,1,-2)$-SUM $\equiv$ AVERAGE

idea (Dudek-Gawrychowski-Starikovskaya : STOC'20)
make each set 3-avg-free.

Behrend's Thm (from additive combinatorics, 1940's):
$\exists\, S \subseteq [-U, U]$ of size $\Omega\left(\frac{U}{2^{\sqrt{\lg U}}}\right)$ s.t. S is average-free

no $x, y, z \in S$
s.t. $\frac{x+y}{2} = z$

for full reduction see [DGS'20, section 3]

more easy equivalences:

Target-3SUM : $\exists\ a \in A, b \in B, c \in C$
$a+b+c = t$?

Positive-3SUM' : $\exists\ a \in A, b \in B, c \in C$, $A,B,C \subseteq [0,+U]$
$a+b=c$

Small-Integer-3SUM: given $S \subseteq [-n^{3.1}, n^{3.1}]$ ....

Thm: 3SUM $\rightarrow$ Small-Int-3SUM

pf: - repeat $10 \lg n$ times:
- pick uniformly random prime in $[2, n^{3.1}]$
- $S' = \{ x \bmod p \mid x \in S \}$
- if Small-Int-3SUM(S') = no, return no.
- return yes.

targets: $0, p, 2p, -p, -2p$.

time: ✓

correctness: yes ⇒ yes ✓
no ⇒ no w.h.p.

if $a+b+c \neq 0 \Rightarrow$

$c(\lg n)$

$$\Pr[a+b+c=0 \bmod p] = \frac{[\#\text{ prime divisors of } a+b+c \text{ in } [2, n^{3.1}]]}{\#\text{ of primes in } [2, n^{3.1}]}$$

$$\Omega\left(\frac{n^{3.1}}{\log n}\right)$$

$$= O\left(\frac{\log n}{n^{3.1}}\right)$$

$\Rightarrow$ Union Bound: $\Pr[\exists a,b,c \in S : a+b+c=0 \bmod p]$

$$\le \frac{n^3}{n^{3.1}/\log n} < 1/10$$

$\Rightarrow$ if $3SUM(S) = no$: $\Pr[\forall S' : \text{small-}3SUM(S') = yes]$

$$\le \left(\frac{1}{10}\right)^{10 \log n} < \frac{1}{n^{10}}$$

<u>Strong 3SUM Conj</u>: no $O(n^{2-\varepsilon})$ alg if $U = n^2$.

- $O(n + U \log U)$ alg. [next week.]