Periods and L-functions

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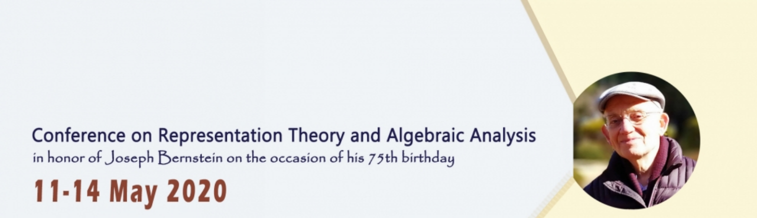

A sequence of periods

Let $X^\bullet$ be the quotient of $(\mathrm{SL}_2)^n$ by the subgroup H_n, where:

$$H_n = \left\{ \begin{pmatrix} 1 & x_1 \\ & 1 \end{pmatrix} \times \begin{pmatrix} 1 & x_2 \\ & 1 \end{pmatrix} \times \cdots \times \begin{pmatrix} 1 & x_n \\ & 1 \end{pmatrix} \middle| x_1 + x_2 + \cdots + x_n = 0 \right\}.$$

It has an action of $G = (\mathbb{G}_m \times (\mathrm{SL}_2)^n)/\pm 1$, and corresponds to the following periods of automorphic forms $f \in \pi$ on $[G] := G(k)\backslash G(\mathbb{A})$:

- $n = 1$, Hecke: $\int_{[\mathbb{G}_m]} f \begin{pmatrix} a & \\ & 1 \end{pmatrix} |a|^s da$, represents $L(\pi, \frac{1}{2} + s)$.
- $n = 2$, Rankin–Selberg: $X^\bullet \hookrightarrow \mathbb{A}^2 \times^{\mathrm{GL}_2} G$, $\Phi \in \mathcal{S}(\mathbb{A}_2)$, $\int_{[\mathrm{GL}_2]} f_1(g) f_2(g) E_\Phi(g, \frac{1}{2} + s) dg$, represents $L(\pi_1 \times \pi_2, \frac{1}{2} + s)$.
- $n = 3$, Garrett: $X^\bullet \hookrightarrow [S, S]\backslash \mathrm{Sp}_6$, $\int_{[G/\mathbb{G}_m]} f(g) E_{\mathrm{Siegel}}(g, \frac{1}{2} + s) dg$, represents $L(\pi_1 \times \pi_2 \times \pi_3, \frac{1}{2} + s)$.

To fix ideas:

⋆All formulas approximate, Archimedean places omitted!⋆

$$[H]/K_H(=\mathrm{Bun}_H) \to [G]/K_G(=\mathrm{Bun}_G), \;\; K_G = G(\widehat{\mathcal{O}})$$

For f on $[G]/K_G$,

$$\int_{[H]} f dh = \int_{[G]} f(g)\cdot 1_{[H]K_G}(g)$$

The "period distribution" $1_{[H]K_G}$ is the image of

$$1_{H\backslash G(\widehat{\mathcal{O}})} \in \mathcal{S}(H\backslash G(\mathbb{A}))$$

under the "theta series"

$$\Phi \mapsto \Theta\Phi(g) := \sum_{\gamma\in H\backslash G(k)} \Phi(\gamma g).$$

Moral: The period distribution coming from $1_{X^\bullet(\widehat{\mathcal{O}})}$ (for $X^\bullet = H\backslash G$) may be wrong!

Basic functions

Braverman–Kazhdan: Define a non-trivial Schwartz space for the "basic affine space" $X = \overline{N\backslash G}^{\text{aff}} = \operatorname{spec} k[G]^N$, generalizing:
$G = \mathrm{GL}_2$, $X^\bullet = N\backslash \mathrm{SL}_2 = \mathbb{A}^2 \smallsetminus \{0\} \hookrightarrow X = \mathbb{A}^2$.
At any finite place,

$$X^\bullet(\mathfrak{o}) = \{(x,y) \in \mathfrak{o}^2 | (x,y) = 1\}, \;\; X(\mathfrak{o}) = \mathfrak{o}^2.$$

$\mathcal{S}(X^\bullet(\mathbb{A})) \xrightarrow{\int_{\mathbb{A}^\times} \chi^{-1}} \mathrm{Ind}_B^G(\chi) \xrightarrow{\mathcal{E}} C^\infty([G])$ Eisenstein series
Difference between $\mathcal{S}(X^\bullet(\mathbb{A}))$ and $\mathcal{S}(X(\mathbb{A}))$ is

$$E(z,s) = \sum_{(m,n)=1} \frac{y^s}{|mz+n|^{2s}} \;\text{ vs. }\; E^*(z,s) = \sum_{(m,n)\neq(0,0)} \frac{y^s}{|mz+n|^{2s}}$$

Here, $E^*(z,s) = \zeta(2s)E(z,s)$, both have meromorphic continuation.

$X = \overline{N\backslash G}^{\text{aff}}$

$\mathcal{S}(\overline{N\backslash G}^{\text{aff}})$ originates in *Drinfeld's compactification* of Bun_B (=global model for $N\backslash G(\mathfrak{o})$).

$$\overline{\mathrm{Bun}_B} = \mathrm{Maps}(C, \overline{N\backslash G}^{\text{aff}}/T \times G)$$

$\overline{\mathrm{Bun}_B}$ is singular, so we want to compute $IC_{\overline{\mathrm{Bun}_B}}$.
Function-theoretically (taking Frobenius trace):

$$\mathcal{S}(X(\mathbb{A})) \ni IC_{X(\widehat{\mathcal{O}})},$$

where for $\mathfrak{o} = \mathbb{F}((t))$ we'll think of $X(\mathfrak{o})$ as the $\mathbb{F}$-points of the *arc space* of X: $\mathcal{L}^+X = \mathrm{Maps}(D = \mathrm{spec}\,\mathbb{F}[[t]], X)$.

$$IC_{\overline{\mathrm{Bun}_B}} = \Theta\left(IC_{X(\widehat{\mathcal{O}})}\right) \in C^\infty([T] \times [G]).$$

Braverman–Finkelberg–Gaitsgory–Mirković [BFGM]:

> *"The Eisenstein series $E^*(g,\chi) = \int_{[T]} IC_{\overline{\mathrm{Bun}_B}}(t,g)\chi^{-1}(t)dt$ attached to $\overline{\mathrm{Bun}_B}$ is $\prod_{\check\alpha>0} L(\chi\circ\check\alpha, 0)$ times the Eisenstein series $E(g,\chi)$ attached to Bun_B."*

Back to $X^\bullet = H\backslash G$. Idea:

- Choose an affine $X^\bullet \hookrightarrow X$ (e.g., $X = \overline{X^\bullet}^{\text{aff}}$).
- Define a Schwartz space $\mathcal{S}(X(\mathbb{A}))$ with a "basic vector" $\Phi^0 = IC_{X(\hat{\mathcal{O}})}$.
- Define the "X-period" as the theta series $P_X(g) = \Theta\Phi^0(g) = \sum_{\gamma \in X(k)} \Phi^0(g)$.

Conjecture (S., 2009)

For $f \in \pi^{G(\hat{\mathcal{O}})}$ an automorphic form, suitably normalized (e.g., by Fourier–Whittaker coefficient), $\int_{[G]} f \cdot P_X$ is equal to (a special value of) an L-function.

The case of L-monoids

Example: $X^\bullet = H = \mathrm{GL}_n \hookrightarrow X = \mathrm{Mat}_n$, this unfolds to the Godement–Jacquet integral (here $G = H \times H$, $f = \phi \otimes \bar{\phi}$):

$$\int_{[H]} \langle \pi(h)\phi, \phi \rangle \, \Phi^0(h) dh = L(\pi, -\frac{1}{2}(n-1))$$

For (split) $H \xrightarrow{\det} \mathbb{G}_m$, and any section $\lambda : \mathbb{G}_m \to H$, identified with a heighest weight for the dual group $\check{H}$, Braverman–Kazhdan and Ngô defined an L-monoid $H \hookrightarrow H_\lambda$, generalizing $\mathrm{GL}_n \hookrightarrow \mathrm{Mat}_n$.
Bouthier–Ngô–S. (2014):

1. For $\mathfrak{o} = \mathbb{F}[[t]]$, the function $IC_{X(\mathfrak{o})}$ is well-defined. (Rests on the Grinberg–Kazhdan–Drinfeld theorem on finite-dimensionality of singularities of $\mathcal{L}^+X$.)
2. The GJ integral in this case gives $L(\pi, V_\lambda, -\langle \rho, \lambda \rangle)$.

The general case

What is the L-value attached to a general spherical X?
The Gaitsgory–Nadler dual group: $\check{G}_X \hookrightarrow \check{G}$. (Conjecturally, only automorphic forms with Langlands parameters into $\check{G}_X$ have nonzero pairing with P_X.)
E.g., in the group case that we just saw $X = H$, $G = H \times H$, and the representation must be of the form $\pi \otimes \tilde{\pi}$, so $\check{G}_X = \check{H}$.
Looking for a representation $\rho_X : \check{G}_X \to \mathrm{GL}(V)$.

S. (2009) Let $X = H\backslash G$ with H reductive, so $IC_{X(\mathfrak{o})} = 1_{X(\mathfrak{o})}$. General formula in terms of the *colors* of X:

$$X /\!\!/ N \curvearrowleft T = B/N, \text{ acts through some quotient } T \twoheadrightarrow T_X.$$

The toric variety $X /\!\!/ N$ defines certain coweights of T_X which are weights of the representation ρ_X of $X /\!\!/ N$.

Examples

- $X = H \curvearrowleft G = H \times H$. The colors are the Bruhat divisors, inducing valuations equal to the simple coroots $\check{\alpha}$. Hence $\rho_X = \check{\mathfrak{h}}$. Corresponds to Petersson diagonal period of normalized cusp forms:
$$\int_{[H]} f(h)\overline{f(h)}dh = L(\pi, \mathrm{Ad}, 1)$$
- $X = \mathrm{PGL}_2^{\mathrm{diag}} \backslash \mathrm{PGL}_2^3$. B-orbits $\leftrightarrow$ PGL_2-orbits on $(\mathbb{P}^1)^3 \ni (z_1, z_2, z_3)$, colors = $\{z_i = z_j\}$, valuations $\frac{\check{\alpha}_i + \check{\alpha}_j - \check{\alpha}_k}{2}$.
Hence, $\rho_X = \mathrm{Std} \otimes \mathrm{Std} \otimes \mathrm{Std}$ of SL_2^3.
The (Kudla, Harris, Gross, Böcherer, Schulze-Pillot, Watson, Ichino) triple product period
$$|\int_{[\mathrm{PGL}_2]} f_1(g) f_2(g) f_3(g) dg|^2 = L(\pi_1 \times \pi_2 \times \pi_3, \frac{1}{2}).$$

S.–Jonathan Wang (2020?) Vast generalization of the above result and Bouthier–Ngô–S. (for $\check{G}_X = \check{G}$, for now): Description of $IC_{X(\mathfrak{o})}$ in terms of the geometry of X.

Example: Let $X^\bullet = H_n \backslash G_n$ the $(\mathbb{G}_m \times \mathrm{SL}_2^n)/\pm 1$-variety from the beginning of this lecture, $X = \overline{X^\bullet}^{\mathrm{aff}}$. Then, for a cusp form $f_s \in \pi = |\bullet|^s \otimes \pi_1 \otimes \cdots \otimes \pi_n$, $\Re(s) \gg 0$, Whittaker coefficient 1,

$$\int_{[G]} f \cdot \Theta(IC_{X(\widehat{\mathcal{O}})}) = L(\pi_1 \times \cdots \times \pi_n, \frac{1}{2} + s).$$

Writing $\int_{[G]} f \cdot P_X$ as an Euler product is
sometimes easy (Godement–Jacquet, Hecke, Rankin–Selberg), and sometimes hard (Gan–Gross–Prasad).

The Ichino–Ikeda conjecture:
For $X = H\backslash G = \mathrm{SO}_n \backslash (\mathrm{SO}_n \times \mathrm{SO}_{n+1})$,

$$\left| \int_{[G]} f \cdot P_X \right|^2 = \left| \int_{[H]} f(h) dh \right|^2 = \int_{H(\mathbb{A})} \langle \pi(h) f, f \rangle \, dh \text{ (an Euler product).}$$

The observation of Akshay Venkatesh: The RHS is equal to the Plancherel density of $1_{X(\mathfrak{o})}$ (when f is everywhere unramified).
Hence, the L-value attached to the period can be computed by a local Plancherel formula:

$$\|1_{X(\mathfrak{o})}\|^2_{L^2(X(F))} = \int_{\hat{G}^{unr}} L(\pi, \rho_X) \mu(\pi)$$

Of course, *there is no a priori reason why the Plancherel formula should involve an L-function*! A priori, what is denoted by $L(\pi, \rho_X)$ could be any function of π.

In any case, generalizing the Ichino–Ikeda conjecture (S.–Venkatesh), we have a path from periods to L-functions:

$$\left| \int_{[G]} f \cdot P_X \right|^2$$

II conjecture $\rightsquigarrow$ Euler product of Plancherel densities of $\| IC_{X(\mathfrak{o})} \|^2_{L^2(X(F))}$

local calculation $\rightsquigarrow$

$$\int_{\hat{G}^{unr}} L(\pi, \rho_X) \mu(\pi).$$

The aforementioned [S. 2009, S.–Wang 2020?] perform this local calculation.

Why do L-functions appear in the Plancherel decomposition of $\|IC_{X(\mathfrak{o})}\|^2_{L^2(X(F))}$?
$F = \mathbb{F}((t)) \supset \mathfrak{o}$. Let's think of elements Φ_i of $S(X(F))^{G(\mathfrak{o})}$ as Frobenius traces of $\mathcal{L}^+G$-equivariant ℓ-adic sheaves $\mathcal{F}_i$ on the loop space $\mathcal{L}X$; then

$$\langle \Phi_1, \Phi_2 \rangle = \mathrm{tr}(\mathrm{Frob}, \mathrm{Hom}(\mathcal{F}_1, D\mathcal{F}_2)),$$

(all objects and Homs in the derived category).
We are led to study $D(\mathcal{L}X/\mathcal{L}^+G)$, e.g., $X = H, G = H \times H$, this is the bounded derived category of $H(\mathfrak{o})$-equivariant constructible sheaves on the affine Grassmannian of H.
Bezrukavnikov–Finkelberg: There is a natural equivalence of triangulated categories ("the spherical category"):

$$D(\mathcal{L}^+H \backslash \mathcal{L}H/\mathcal{L}^+H) \xrightarrow{\sim} Coh_{\mathrm{perf}}(\check{\mathfrak{h}}^*/\check{H}) = Coh_{\mathrm{perf}}(T^*\check{H}/(\check{H} \times \check{H})).$$

Notice that the (co)adjoint representation $\check{\mathfrak{h}}^*$ is the one that shows up in the calculation of the Petersson norm=diagonal period of normalized cusp forms:

$$\int_{[H]} f(h)\overline{f(h)}dh = L(\pi, \mathrm{Ad}, 1)$$

Conjecture (Ben Zvi–Venkatesh–S.)

Given a spherical variety X, there is a Hamiltonian $\check{G}$-space $\check{M} \to \check{\mathfrak{g}}$ and an equivalence of module categories for the spherical category:

$$D(\mathcal{L}X/\mathcal{L}^+G) \xrightarrow{\sim} Coh_{perf}(\check{M}/\check{G}).$$

Moreover, $\check{M} = V_X \times^{\check{G}_X} \check{G}$, where $\rho_X : \check{G}_X \to \mathrm{GL}(V_X)$ is the representation attached to the L-value of X.
The association $M = T^*X \leftrightarrow \check{M}$ is involutive, i.e., if $\check{M} = T^*\check{X}$, the Hamiltonian dual of $\check{X}$ is M.

Examples

M	$\check{M}$
T^*H $\int_{[H]^{\text{diag}}} f_1(h)f_2(h)dh = L(\tau, \text{Ad}, 1), \pi = \tau \otimes \tilde{\tau}$	$T^*\check{H}$
$T^*((N,\psi)\backslash G) = (\mathfrak{t}^* /\!\!/ W) \times G$ (Whittaker normalization)	pt$= \check{G}/\check{G}$ $\int_{[\check{G}]} 1 = L(\mathfrak{t}^* /\!\!/ W)$ (motive of G)
Tate: $T^*\mathbb{A}^1$ $\int_{[\mathbb{G}_m]} \chi(x)\Theta\Phi(x)dx = L(\chi, 0)$	$T^*\mathbb{A}^1$
Hecke: $T^*(\mathbb{G}_m\backslash \text{PGL}_2)$ $\int_{[\mathbb{G}_m]} f\begin{pmatrix} a & \\ & 1\end{pmatrix} da = L(\pi, \text{Std}, \frac{1}{2})$	$T^*\text{Std} \curvearrowleft \text{SL}_2$ $(E^*(g,\chi) = L(\chi \circ \check{\alpha}, 0)E(g,\chi))$
Gross–Prasad: $T^*(\text{SO}_{2n}\backslash\text{SO}_{2n} \times \text{SO}_{2n+1})$ $\lvert \int_{[H]} f_1(h)f_2(h)dh\rvert^2 = L(\pi_1 \times \pi_2, \frac{1}{2})$	Theta: $(W_{4n}, \omega) \curvearrowleft \text{SO}_{2n} \times \text{Sp}_{2n}$. Rallis inner product formula.

Remark: All these examples are spherical/multiplicity-free and smooth affine. Other examples suggest that losing the spherical property on one side leads to singularities/stacky behavior on the other.

Open problems

1. Proving the geometric conjecture: Recent new cases by Braverman–Finkelberg–Ginzburg–Travkin (unramified), Raskin. Can we upgrade the calculation of the Plancherel formula (with J. Wang) to the conjectural categorical equivalence?
2. Euler factorization and relative functoriality: it could be possible to establish the Ichino–Ikeda formula through a comparison of relative trace formulas:
$$\mathcal{S}(X \times X/G) \xrightarrow{\sim} \mathcal{S}(N, \psi \backslash G/N, \psi).$$
The geometric framework suggests the existence of natural such transfer map, corresponding to the pushforward $\check{M} \to \mathrm{pt}$.
3. Functional equation: There should be a Fourier transform $\mathcal{S}(X) \xrightarrow{\sim} \mathcal{S}(X^*)$, where X^* is X with G-action twisted by Chevalley involution, such that the theta series satisfy the Poisson summation formula. Also suggested by the geometric conjectures.