Infinite sums of L-functions
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In this context, all of us encountered curious behavior of certain (usually) infinite sums of L-functions. I will explain this in *Part 1* of the talk.

- *Part 2* – Relative Langlands duality (joint work in progress with Ben-Zvi, Sakellaridis).

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- *Part 3* – indicate how infinite sums of L-functions arise in relative Langlands duality.

I hope that eventually (but not yet!) relative Langlands duality will lead to a much better understanding of *Part 1*.
Throughout I have suppressed many technical details in the statements.

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- For example, 1989 N. V. Kuznetsov discovered a remarkable symmetry, which (informally) says

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$$\sum_{\varphi \in \mathrm{Aut}(\mathrm{GL}_2)} L_\varphi(z_1) L_\varphi(z_2) L_\varphi(z_3) L_\varphi(z_4)$$

is invariant under permutations by $z_i \mapsto Z - z_i$, with $Z = \frac{z_1+z_2+z_3+z_4}{2}$.

- Both P. Michel and I and Bernstein-Reznikov gave more transparent arguments for this formula. Following Reznikov we first describe the corresponding phenomenon in representation theory where it may be more familiar for the current audience.

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- More intrinsically (write $\langle X, Y\rangle = \mathrm{Hom}(X, Y)$):

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- Get an *nontrivial* isomorphism between these sums of lines; the transition matrix is the $6j$ symbol.

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- Before we come to this, we say the abstract principle behind the computation: Restrict from $SU(2)^4$ to $SU(2)$ in stages by first passing to the intermediate subgroup

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- Both inclusions here have *multiplicity one branching.* The same principle applies in many other instances.

the Gelfand-Tsetlin basis

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- Intrinsically:

$$V = \bigoplus_{W_i \in \mathrm{Irr}(U(i))} \mathrm{Hom}(W_1, W_2) \otimes \mathrm{Hom}(W_2, W_3) \otimes \cdots \otimes \mathrm{Hom}(W_{n-1}, V).$$

Each summand is one-dimensional and nonzero precisely when the weights interlace.

- Same principle applies in the obvious way to other situations. We will encounter later the following one: cmpute restriction of a $U(n)$-representation to the torus by successively restricting along

$$\begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \subset \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} \subset \mathrm{U}_3.$$

Automorphic story

Rest of the talk: G is a reductive group over a global field, e.g. GL_n over $\mathbb{Q}$; G_F are its points over some local field, e.g. $\mathrm{GL}_n(\mathbb{R})$.

- automorphic form φ_G (for G) is an eigenfunction of Hecke/differential operators on locally symmetric space $[G]$.

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  Similarly, H_F-invariants on a G_F-representation $\leftrightarrow \int_{[H]} \varphi_G$.
- It is expected that if branching from G to H is multiplicity one then the m_i (or their squares) are L-function values.

- For $\mathrm{GL}_2 \subset \mathrm{GL}_2^4$ we compute $\int_{[\mathrm{GL}_2]} \varphi_1\varphi_2\varphi_3\varphi_4$ by splitting into pairs and decomposing, we get the two sides of KUZNETSOV'S formula via two chains.

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- More generally, when one can link $H \subset G$ by a multiplicity one branching chain, then one often gets

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- We can now produce many interesting identities involving infinite sums of L-functions. (REZNIKOV).

Relative Langlands duality

- For exposition will use the TQFT metaphor for Langlands, first suggested by KAPRANOV. In this metaphor, the study of periods corresponds to the theory of boundary conditions for TQFT. I know very little about TQFT, and I apologize if I use the metaphor ineptly.

Langlands program: global, geometric global, local, geometric local

Manifold	Dimension	What we study
ring of integers e.g. $\mathbf{Z}$	3	vector space functions on $G_{\mathbf{Z}}\backslash G_{\mathbb{R}}$
curve over $\overline{\mathbb{F}}_p$ Σ	2	category of sheaves on$\mathrm{Bun}_G(\Sigma))$
local field F	2	category of G_F-representations
function field e.g. $\mathbf{C}((t))$	1	2-category $G(\mathbf{C}((t))$-categories

The Langlands program posits a description of everything here (together with their symmetries) in terms of a dual picture involving $G^\vee$.

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- In the TQFT metaphor, the extra data is akin to choosing a bounding manifold.

The theory of periods attached to X, a G-variety

ring of integers e.g. $\mathbf{Z}$	fns on $G_{\mathbf{Z}}/G_{\mathbb{R}}$ $\ni$ the X-Poincaré series P_X
curve over $\overline{\mathbb{F}_p}$ Σ	sheaves on $\mathrm{Bun}_G(\Sigma)$ $\ni$ the X-Poincaré sheaf
local field F	category of G_F-rep. $\ni$ Functions(X_F)
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- Taking $X = G/H$ we recover various structures from Part 1.
- Don't expect general nice 'dual" descriptions ... better in the *multiplicity one cases*.

Relative Langlands duality (w/ Ben-Zvi, Sakellaridis)

- *Multiplicity one case (+technical assumptions)*: we give a recipe for a $G^\vee$-space $X^\vee$ and we expect $X^\vee$ controls the dual answer at each level of the table.

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- The physical analogue, S-duality of boundary conditions, has been studied by Gaiotto and Witten.

Back to infinite sums of L-functions (speculative)

A multiplicity one X may dualize to $X^\vee$ that doesn't have multiplicity one. (in which case $X^{\vee\vee}$ is not defined by our recipe).
The corresponding period can often be expressed as an infinite sum of L-functions.
I talk only about only one example.

Example: basic affine space

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- Duality recipe gives $G^\vee = \mathrm{PGL}_n, X^\vee = \mathrm{PGL}_n/T$ and predicts:
- $$\langle P_X, \varphi_G \rangle \sim \sum_{\text{fixed points } x \in X^\vee} L\text{-function for } T_x(X^\vee). \qquad (2)$$

  $$\langle P_{X^\vee}, \varphi_{G^\vee} \rangle \overset{?}{\sim} \sum_{\text{fixed points } x \in X} L\text{-function for } T_x(X). \qquad (3)$$

  First is standard. The second is not proved, but I expect a version of it can be established with suitable regularizations ("cuspidal part" of both side match; Eisenstein story must be analyzed).

- Replace $\mathrm{SL}(V)/U$, which parameterizes flags in V where each subspace W comes with an orientation, by the smooth stack (cf. Laumon compactifiction) parameterizing

$$\text{oriented line} \to \text{oriented plane} \to \ldots \text{oriented } n-1\text{- space} \to V.$$

(reminiscent of G-T!)

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- We want to check:

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- LHS $= \int_{[\text{torus}]} \varphi_{\mathrm{PGL}_n}$ is an infinite sum of L-functions via multiplicity one chain from torus and PGL_n.

$$\mathrm{LHS}_{cusp} = \mathrm{RHS}_{cusp}, \mathrm{LHS}_{Eis} \stackrel{?}{=} \mathrm{RHS}_{Eis}.$$

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- A powerful method to analyze such situations is to use multiplicity one branching chains. But as we saw in Part 1, there may be more than one such chain. It is crucial to understand better how this fits with the duality paradigm.
- Happy Birthday, Joseph, and thank you for your inspiring mathematics!