Unipotent characters and \(\ell\)-adic sheaves.

Goal: \underline{geometric} way to write characters of \(p\)-adic groups.

Recall the answer for finite Chevalley groups.

Lusztig's theory of Character sheaves. \(G\) - reductive over \(\mathbb{F}_q\).

1) \(G = G(\mathbb{F}_q)\)

Then to an \(\mathfrak{g}\) - \(\mathfrak{u}\) repn of \(G\) there corresponds an irreducible perverse sheaf on \(\mathfrak{g}\), \(\mathcal{F}_g \in \text{Perv}_{\mathfrak{c}}(\mathfrak{g})\)

\[ D_{\mathfrak{c}}(G) \]

\[ X_g(g) = T_{2}(\mathbb{F}_2, \mathcal{F}_g|_{g}) \]
\[ X_g = \text{Tr}_2 (F_2, F_g). \]

Example: \[ g \in \mathbb{C}[G/B]. \]

Such \[ g \] with ir. reps of \( H = \mathbb{C}[B \setminus G/B] \):

\[ g \in \mathcal{S}_n \in \mathcal{S}_{n!}. \]

\[ \mathcal{F}_g = \text{Sp}_2 \otimes \mathbb{C}[W]. \]

\[ \text{Sp}_2 = \prod_k \overline{\mathbb{Q}_E} [d]. \]

\[ \pi: \tilde{G} \longrightarrow G \]

\[ \text{constant sheaf} \quad G/B \times G \]
16) \( G \) - any (connected center)

There is an assignment \( \mathfrak{g} \sim \mathfrak{g}_G \rightarrow \mathfrak{g}_\mathfrak{g} \) - irreducible powers sheaf

\[ \text{In} \langle \mathfrak{g} \rangle \text{ (a character sheaf)} \]

\[
\begin{align*}
X_g & \hookrightarrow \text{Tr} (F_G, F_g) \\
\text{In} \langle \mathfrak{g} \rangle & = \sim \text{families}, \text{ for each family } R
\end{align*}
\]

one defines a finite group \( \Gamma = \Gamma_R \).

\[
\begin{align*}
s.t. \ R & \leftrightarrow (\text{In} \mathfrak{g}, \mathfrak{g}_G (\Gamma)) \Rightarrow \{ (\gamma, \psi) \mid \gamma \in \Gamma, \ \psi \in \text{In} \langle \mathfrak{g} (\Gamma) \rangle \} \\
\text{Rank} \ K (\mathfrak{g}_G (\Gamma)) & \sim \text{Fun} (\mathfrak{g}_G (\Gamma), \mathfrak{g} (\Gamma)) \ni \chi \mid \chi_1, \chi_2 \in \chi, \chi_1 \otimes \chi_2 \end{align*}
\]

\[
\begin{align*}
\text{Comm} (\Gamma) & = \{ (\delta_1, \delta_2) \in \mathfrak{g}^2 \mid \delta_1, \delta_2 = \delta_2, \delta_1 \} \\
\mathfrak{g} & \sim \mathfrak{g} (F_G, F_g).
\end{align*}
\]
The transformation matrix between $x_g$, $T_{21} F_2 F_3$ is the matrix of $\Phi$.

\[ S_z = \text{Rep}(\Gamma \times \Gamma) \cong F_T. \]

self-dual abelian group.

\[ \text{Rmk 1) } CS \cong Z(\text{H}_f), \quad \text{H}_f - \text{finite Hecke category} \]

Driinfeld center on $3 \times 3$ monodromic sheaves on $G$.

\[ \text{H}_f = D(\beta : G : B) \]

Ben-Zvi - Nadler

Finkelberg, Ostrik

Levendorskii

in different contexts this is due to

Leustig.
Rmk 2. \[ C[G]^g = C[G]^g \]

\[ \mathcal{Z}(C[G]) \]

Center won't be true for p-adic groups.

2) Change notation! \[ G = G \left( \mathbb{F}_q(\mathbb{H}) \right) = G(\mathbb{F}_q) \]

2a) \[ G = GL_n. \]

\[ g - \text{a 1r irreducible) representation of } G \geq \mathbb{F}(0) \geq I - \text{Iwahori} \]

\[ g^I \neq 0, \text{ } g \text{ is generated by } g^I. \]

\[ G = G_c - \text{union of compact subgroups.} \]
Then $\chi_p |_{G_w} = T_2(F_2, \mathbb{Sp}_2 \otimes \overline{\mathbb{G})}$. 

$g \leftrightarrow \text{rep of } H_{aff} = \mathbb{C}((\mathbb{F}/I) G/I) \rightarrow \mathbb{C}[W_{aff}]$

$\overline{g}$ - degeneration of $g$.

Idea of proof:

It's enough for every parahoric $P$ to prove equality after restricting to $P$ and then also after taking push-forward to $\overline{P} = P/\text{prinmp. radical}$. - reduces to the f. dem. statement.
2 b) \( G \) - general (split),

If \( H \) is a (reductive) group \( / C \), \( \text{Comm}(H) \) - variety of comm-Hg pairs.

\[
O(\text{Comm}(H))^H \supset O_e(\text{Comm}(H))^H = \{ f \mid \forall y, f|_{x(y)x^(-1)} \text{ is constant} \}
\]

\[
O_2 = \{ f \mid \forall X, f|_{x X x} \}
\]

\[
O_{e2} = O_e \cap O_2.
\]

- Recall \( G \) - \( p \)-adic group. \( S = C_e^\infty(G) \)

\[
C(S) = S / [S, S]
\]

\[
\text{Dist}(G) = C(S)^*.
\]

\[
\text{Rep}(G) = \text{Rep}(G)_u \oplus \text{Rep}(G)_n.
\]

\[
S = S_u \oplus S_{nu}, \quad \Rightarrow \quad C(S) = C(S)_u \oplus C(S)_n.
\]

\( \text{Rep} G_u = \{ \rho \mid g^2 \neq 0 \} \& \text{their } L \text{-packets} \)
\[ G = G_c \sqcup G_{nc} \]
\[ S(G) = S(G_c) \oplus S(G_{nc}) \]
\[ C(S) = S_G = C(S)_c \oplus C(S)_{nc} \]

**Lemma**
\[ C(S)_u = C(S)_{uc} \oplus C(S)_{unc} \]

**Conj**
There are canonical isomorphisms:
1. \[ \hat{C}_{weu} = \oplus_{u \in U/\hat{G}} \mathcal{O} \left( \text{Comm} \mathbb{Z}(\hat{G})^{\text{red}} \right) \]
2. \[ \hat{C}_{weu} = \oplus \mathcal{O}_{\mathbf{Z}_2} \]

This agrees with characters & almost characters.
Unipotent reps $\rightarrow (u, s, \psi)$

$u s = s u, s^4 u \in G$

$\psi \in \text{Im Rep}(\Pi_0(\mathbb{Z}(s, u)))$

$G$ is simple, $u$ is unipotent.

$\mathfrak{g} u/s, q - \text{standard rep. of } n.$

$\chi_{[u/s, q]} : \mathbb{C} \rightarrow \mathbb{C}$,

$f \rightarrow \langle f_u | \mathbb{Z}(s), s, \psi \rangle.$

$C_{u, c}$ has an obvious involution.

This sends character to almost character - $f \rightarrow$
coming from a character sheaf.  

Rmk. Partly inspired by Lusztig's paper on unpotent \( \mathfrak{g} \)-on loop groups.

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\[ H^\omega = \mathcal{D} \text{Coh} \mathcal{C}'(\text{g}_F) \]

\[ \mathcal{L}(H) = \mathcal{D} \text{Coh} (\text{Comm} (\mathcal{C})) \]

\[ \mathbb{P} \rightarrow F_{5,n,1,4} \]

\[ \psi \in \mathfrak{t} \]
Conj b) \[ \Rightarrow \] \( C_{u, c} \) carries an involution \( \phi \).

Idea: \( \phi \): characters of standard model \( G \) to almost character \( f-n \) coming from \( CS \).

Rank Bouthier-Kazhdan-Varshavsky

define \( \text{Per}^{G}(\xi) \)

we expect these \( CS \) to lie in \( \text{Per}^{G}(\xi) \)

with Varshavsky have a similar result for cuspidal depth 0 \( L \)-packets.
for cuspidal depth 0 L-packets.