SECOND ADJOINTNESS VIA NEARBY CYCLES: A REPORT ON THE WORK OF LIN CHEN

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1. Second adjointness for $\mathfrak{p}$-adic groups

1.1. **Notations.** We let G be a split reductive group, with a chosen Cartan subgroup $T \subset G$, and a polarization, so that we have the well-defined positive and negative Bore subgroups

$$B^- \supset T \subset B.$$

Let $N \subset B$ and $N^- \subset B^-$ denote their unipotent radicals.

In this section, we will use boldface symbols such as $\mathbf{G}$, $\mathbf{T}$, etc., to denote the points of the corresponding group over a $\mathfrak{p}$-adic field $\mathbf{K}$.

1.2. **Jacquet and induction functors.**

1.2.1. Consider the $(\mathbf{G}, \mathbf{T})$-bimodules

$$\mathrm{Funct}_c(\mathbf{G}/\mathbf{N}) \text{ and } \mathrm{Funct}_c(\mathbf{G}/\mathbf{N}^-), \tag{1.1}$$

1.2.2. Given a bimodule $Q \in (\mathbf{G}_1 \times \mathbf{G}_2)$-mod, we can define functors

$$F_{Q,1\to 2} \text{ and } F_{Q,2\to 1}$$

by

$$F_{Q,1\to 2}(M_1) = Q \underset{\mathcal{H}(\mathbf{G}_1)}{\otimes} M_1 \text{ and } F_{Q,2\to 1}(M_2) = Q \underset{\mathcal{H}(\mathbf{G}_2)}{\otimes} M_2,$$

respectively, where $\mathcal{H}(\mathbf{G}_i)$ denotes the Hecke algebra of $\mathbf{G}_i$.

Remark 1.2.3. The functors $F_{Q,1\to 2}$ and $F_{Q,2\to 1}$ are each other's duals for the canonical self-duality on on $\mathbf{G}$-mod, given by

$$M', M'' \in \mathbf{G}\text{-mod} \mapsto M' \underset{\mathcal{H}(\mathbf{G})}{\otimes} M'' \in \mathrm{Vect}\,.$$

The corresponding anti self-equivalence on $\mathbf{G}$-mod^c (the subcategory of compact objects) is the *cohomological duality*

$$M \mapsto \mathcal{H}om_{\mathcal{H}(\mathbf{G})}(M, \mathcal{H}(G)).$$

1.2.4. Consider the functors

$$\mathbf{T}\text{-mod} \leftrightarrow \mathbf{G}\text{-mod}$$

corresponding to the bimodules (1.1).

These are the induction and Jacquet functors,

$$i, i^- : \mathbf{T}\text{-mod} \to \mathbf{G}\text{-mod and } r, r^- : \mathbf{G}\text{-mod} \to \mathbf{T}\text{-mod},$$

respectively.

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1.2.5. Almost by definition, the functor r is given by

$$M \mapsto M_{\mathbf{N}},$$

and similarly for r^-.

Now, the fact that $\mathbf{G}/\mathbf{B}$ is compact allows to rewrite $i(M)$ as

$$\mathrm{Funct}(\mathbf{G}/\mathbf{N}, M)^{\mathbf{T}},$$

which implies that i is naturally the right adjoint of r. And similarly for the pair (r^-, i^-)

Remark 1.2.6. It is more convenient to replace the original (r, i) by their *normalized* versions, by incorporating the ρ-shift. In what follows we will assume having done so.

1.3. **Second adjointness.**

1.3.1. Bersnstein's second adjointness theorem says that there is another, much less obvious adjunction between the above functors. Namely, it says:

Theorem 1.3.2 (Bernstein)**.** *The functors (i^-, r) form an adjoint pair.*

1.3.3. The statement of Theorem 1.3.2 should be complemented by the following:

An adjunction between a given pair of functors is uniquely determined by specifying either the unit or the counit of the adjunction.

In the case of (i^-, r) we a priori specify what the unit of the adjunction is. Note that the composite

$$r \circ i^-,$$

viewed as a functor $\mathbf{T}$-mod $\to$ $\mathbf{T}$-mod is given by the bimodule

$$\mathrm{Funct}_c(\mathbf{N}\backslash\mathbf{G}/\mathbf{N}^-).$$

The unit is given by the map

$$\mathcal{H}(\mathbf{T}) = \mathrm{Funct}_c(\mathbf{T}) \simeq \mathrm{Funct}_c(\mathbf{N}\backslash\overset{\circ}{\mathbf{G}}/\mathbf{N}^-) \to \mathrm{Funct}_c(\mathbf{N}\backslash\mathbf{G}/\mathbf{N}^-), \tag{1.2}$$

where

$$\mathbf{N}\backslash\overset{\circ}{\mathbf{G}}/\mathbf{N}^- \subset \mathbf{N}\backslash\mathbf{G}/\mathbf{N}^-$$

is the big Bruhat cell, and the last arrow in (1.2) is given by extending a (compactly supported function) by zero.

1.3.4. One approach to prove Theorem 1.3.2 is to exhibit explicitly the counit of the adjunction. This approach has been realized by R. Bezrukavnikov and D. Kazhdan in [BK].

Note that the unit is supposed to be a map of $\mathbf{G} \times \mathbf{G}$-modules

$$\mathrm{Funct}_c((\mathbf{G}/\mathbf{N} \times \mathbf{G}/\mathbf{N}^-)/\mathbf{T}) \to \mathrm{Funct}_c(\mathbf{G}). \tag{1.3}$$

This map is constructed using a piece of geometry, which will be crucial throughout this talk.

1.3.5. Choose a dominant regular coweight $\gamma : \mathbb{G}_m \to T$. Consider the resulting adjoint action of $\mathbb{G}_m$ on G. Its attractor/repeller/fixed locus is B, B^- and T-respectively.

1.3.6. More generally, let Y be a smooth affine scheme, equipped with an action of $\mathbb{G}_m$. Let Y^+, Y^-, Y^0 be the corresponding attractor, repeller and fixed point loci. We have the inclusions

$$Y^0 \overset{\mathsf{s}^+}{\to} Y^+ \overset{\mathsf{p}^+}{\to} Y \text{ and } Y^0 \overset{\mathsf{s}^-}{\to} Y^- \overset{\mathsf{p}^-}{\to} Y$$

and the projections

$$Y^+ \overset{\mathsf{q}^+}{\to} Y^0 \overset{\mathsf{q}^-}{\leftarrow} Y^-.$$

In this case, following [DG], we construct the interpolation

$$\widetilde{Y} \subset \mathbb{A}^1 \times Y \times Y,$$

by letting it be the closure along

$$\mathbb{G}_m \times Y \times Y \hookrightarrow \mathbb{A}^1 \times Y \times Y.$$

of the graph of the action map

$$\mathbb{G}_m \times Y \to Y,$$

which is a closed subscheme in $\mathbb{G}_m \times Y \times Y$.

By definition, the fiber of $\widetilde{Y}$ over $1 \in \mathbb{A}^1$ is the diagonal copy $Y \subset Y \times Y$. Furthermore, one can show that the preimage of $0 \in \mathbb{A}^1$ identifies with

$$Y^+ \underset{Y^0}{\times} Y^- \subset Y^+ \times Y^- \subset Y \times Y.$$

Moreover, $\widetilde{Y}$ carries an action of $\mathbb{G}_m \times \mathbb{G}_m$ compatible with the action on $\mathbb{A}^1$ via the multiplication map $\mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$, and the action on $Y \times Y$, given by

$$(c_1, c_2) \cdot (y_1, y_2) = (c_1 \cdot y_1, c_2^{-1} \cdot y_2).$$

1.3.7. Applying this to G with the above action of $\mathbb{G}_m$, we obtain a group-scheme

$$\widetilde{G}^\gamma \to \mathbb{A}^1,$$

whose fiber over $1 \in \mathbb{A}^1$ is the original G, whose whose fiber over 0 is $B \underset{T}{\times} B^-$.

In particular, we can consider the $\mathbb{A}^1$-family of schemes

$$\mathbb{A}^1 \times G \times G/\widetilde{G}^\gamma,$$

equipped with an action of $G \times G$, whose fiber over $1 \in \mathbb{A}^1$ is G and whose fiber over $0 \in \mathbb{A}^1$ is

$$(G/N \times G/N^-)/T.$$

1.3.8. Let us take the set of points of $\mathbb{A}^1 \times G \times G/\widetilde{G}^\gamma$ over our $\mathfrak{p}$-adic field $\mathbf{K}$, i.e.,

$$\mathbf{K} \times \mathbf{G} \times \mathbf{G}/\widetilde{\mathbf{G}}^\gamma. \tag{1.4}$$

This is a space equipped with an action of $\mathbf{G}\times\mathbf{G}$ that interpolates between $\mathbf{G}$ and $(\mathbf{G}/\mathbf{N}\times\mathbf{G}/\mathbf{N}^-)/\mathbf{T}$.

The authors of [BK] use the space of (1.4) to define the operation of *taking the asymptotic*, which is a map

$$\mathrm{Funct}_c((\mathbf{G}/\mathbf{N} \times \mathbf{G}/\mathbf{N}^-)/\mathbf{T}) \to \mathrm{Funct}_c(\mathbf{G}),$$

defining the required map (1.3).

1.4. **The original approach.** Let us briefly describe the original Bernstein's approach to the proof of Theorem 1.3.2.

1.4.1. For a reductive group $\mathbf{G}$, the category $\mathbf{G}$-mod carries another self-duality, called the Cohen-Macaulay duality (also discovered by J. Bernstein). Let us denote the corresponding self-equivalence by

$$\mathbb{D}^{\mathrm{CM}}_{\mathbf{G}} : (\mathbf{G}\text{-mod}^c)^{\mathrm{op}} \to \mathbf{G}\text{-mod}^c.$$

On the subcategory of admissible representations, the Cohen-Macaulay duality is the usual contragredient duality $M \mapsto M^\vee$.

1.4.2. In the same way as one shows that the functor i on admissible representations commutes with contragredient duality, one shows that the following diagram commutes

$$\begin{tikzcd}
(\mathbf{G}\text{-mod}^c)^{\mathrm{op}} \arrow[r, "\mathbb{D}^{\mathrm{CM}}_{\mathbf{G}}"] & \mathbf{G}\text{-mod}^c \\
(\mathbf{T}\text{-mod}^c)^{\mathrm{op}} \arrow[u, "i^{\mathrm{op}}"] \arrow[r, "\mathbb{D}^{\mathrm{CM}}_{\mathbf{T}}"] & \mathbf{T}\text{-mod}^c. \arrow[u, "i"']
\end{tikzcd}$$

Now, it follows formally that the statement of Theorem 1.3.2 is equivalent to the following:

Theorem 1.4.3. *The following diagram of functors commutes:*

$$\begin{tikzcd}
(\mathbf{G}\text{-mod}^c)^{\mathrm{op}} \arrow[r, "\mathbb{D}^{\mathrm{CM}}_{\mathbf{G}}"] \arrow[d, "r^{\mathrm{op}}"'] & \mathbf{G}\text{-mod}^c \arrow[d, "r^-"] \\
(\mathbf{T}\text{-mod}^c)^{\mathrm{op}} \arrow[r, "\mathbb{D}^{\mathrm{CM}}_{\mathbf{T}}"] & \mathbf{T}\text{-mod}^c.
\end{tikzcd} \tag{1.5}$$

For example, for admissible representations, the commutativity in (1.5) means that we have a canonical isomorphism

$$(r(M))^\vee \simeq r^-(M^\vee). \tag{1.6}$$

This will be a prototype of the kind of equivalence we will aim to have in the geometric setting.

2. The geometric setting

2.1. **Notation.** In this section we will working of a ground field k of characteristic 0. Along with our algebraic groups G, T, etc., we will consider the loop/arc groups $\mathfrak{L}^+(G) \subset \mathfrak{L}(G)$, $\mathfrak{L}^+(T) \subset \mathfrak{L}(T)$.

Our basic object of study is categories equipped with an action of $\mathfrak{L}(G)$.

2.2. **Jacquet functors for loop groups.**

2.2.1. First, let $\mathcal{C}$ be a category equipped with an action of a finite-dimensional (or even pro-finite dimensional group H). We can consider the categories of H-invariants and H-coinvariants on $\mathcal{C}$:

$$\mathcal{C}^H \text{ and } \mathcal{C}_H.$$

There is a canonically defined averaging functor

$$\mathrm{Av}^H_* : \mathcal{C} \to \mathcal{C},$$

which canonically factors as

$$\mathcal{C} \to \mathcal{C}_H \to \mathcal{C}^H \to \mathcal{C},$$

and one can show that the resulting functor

$$\mathcal{C}_H \to \mathcal{C}^H \tag{2.1}$$

is an equivalence.

However, this will no longer be the case if we replace H by a group ind-scheme, such as $\mathfrak{L}(N)$.

2.2.2. Let $\mathcal{C}$ be a category acted on by $\mathfrak{L}(G)$. Thus, a feature of the categorical set-up is that we now have two possible Jacquet operations

$$\mathcal{C} \rightsquigarrow \mathcal{C}_{\mathfrak{L}(N)} \text{ and } \mathcal{C} \rightsquigarrow \mathcal{C}^{\mathfrak{L}(N)}.$$

2.2.3. Assume for a moment that $\mathcal{C}$ is dualizable[1]. Then its dual $\mathcal{C}^\vee$ also carries an action of $\mathfrak{L}(G)$.

In this case, we have

$$(\mathcal{C}_{\mathfrak{L}(N)})^\vee \simeq (\mathcal{C}^\vee)^{\mathfrak{L}(N)}. \tag{2.2}$$

[1]Unlike the situation one categorical level down, in which a vector space is dualizable if and only if it is finite-dimensional, most categories we encounter in practice are dualizable; in particular, every compactly generated category is dualizable.

2.3. **Second adjointness for categorical actions?**

2.3.1. We can loosely regard the dualization operation on categories acted on by $\mathfrak{L}(G)$ as an analog of Cohen-Macaulay duality. Then the analogy with Theorem 1.4.3 leads us to the following:

Question 2.3.2. *Is it true that we have a canonical equivalence?*

$$(\mathcal{C}_{\mathfrak{L}(N)})^\vee \simeq (\mathcal{C}^\vee)_{\mathfrak{L}(N^-)}.$$

In fact, Question 2.3.2 had been proposed as a conjecture by Sam Raskin. However, recently, evidence has emerged that the answer in general is "no"; this is an ongoing project by David Yang.

2.3.3. By comparing with (2.2), we can restate Question 2.3.2 (in a more precise form) as follows:

Question 2.3.4. *Is is true that the composite functor*

$$\mathcal{C}^{\mathfrak{L}(N)} \to \mathcal{C} \to \mathcal{C}_{\mathfrak{L}(N^-)} \tag{2.3}$$

is an equivalence?

2.3.5. A partial evidence towards Question 2.3.4 is the following:

Theorem 2.3.6. *The induced functor*

$$(\mathcal{C}^{\mathfrak{L}(N)})^{\mathfrak{L}^+(T)} \to (\mathcal{C}_{\mathfrak{L}(N^-)})^{\mathfrak{L}^+(T)}$$

is an equivalence.

Proof. The proof is inspired by another idea of Joseph Bernstein:

Let $I \subset \mathfrak{L}^+(G)$ be the Iwahori subgroup; let $I_u \subset I$ be its unipotent radical.

One shows, as in the theory of $\mathfrak{p}$-adic groups that the functors

$$(\mathcal{C}^{\mathfrak{L}(N)})^{\mathfrak{L}^+(T)} \hookrightarrow \mathcal{C}^{\mathfrak{L}^+(T)} \overset{\mathrm{Av}_*^{I_u}}{\to} \mathcal{C}^I$$

and

$$\mathcal{C}^I \to \mathcal{C}^{\mathfrak{L}^+(T)} \to (\mathcal{C}_{\mathfrak{L}(N^-)})^{\mathfrak{L}^+(T)}$$

are both equivalences.

We claim that the resulting composite functor

$$(\mathcal{C}^{\mathfrak{L}(N)})^{\mathfrak{L}^+(T)} \hookrightarrow \mathcal{C}^{\mathfrak{L}^+(T)} \overset{\mathrm{Av}_*^{I_u}}{\to} \mathcal{C}^{\mathfrak{L}^+(T)} \to (\mathcal{C}_{\mathfrak{L}(N^-)})^{\mathfrak{L}^+(T)}$$

equals the functor

$$(\mathcal{C}^{\mathfrak{L}(N)})^{\mathfrak{L}^+(T)} \to \mathcal{C}^{\mathfrak{L}^+(T)} \to (\mathcal{C}_{\mathfrak{L}(N^-)})^{\mathfrak{L}^+(T)}.$$

Indeed, let

$$I_u = I_u^+ \cdot I_u^0 \cdot I^+ - u$$

be the triangular decomposition of I_u.

Then the functor

$$(\mathcal{C}^{\mathfrak{L}(N)})^{\mathfrak{L}^+(T)} \hookrightarrow \mathcal{C}^{\mathfrak{L}^+(T)} \overset{\mathrm{Av}_*^{I_u}}{\to} \mathcal{C}^{\mathfrak{L}^+(T)}$$

equals

$$(\mathcal{C}^{\mathfrak{L}(N)})^{\mathfrak{L}^+(T)} \hookrightarrow \mathcal{C}^{\mathfrak{L}^+(T)} \overset{\mathrm{Av}_*^{I_u^-}}{\to} \mathcal{C}^{\mathfrak{L}^+(T)},$$

since the averaging with respect to I_u^0 does nothing on $\mathfrak{L}^+(T)$-invariants, and the averaging with respect to I_u^+ does nothing on $\mathfrak{L}(N)$-invariants.

And the functor

$$\mathcal{C}^{\mathfrak{L}^+(T)} \overset{\mathrm{Av}_*^{I_u^-}}{\to} \mathcal{C}^{\mathfrak{L}^+(T)} \to (\mathcal{C}_{\mathfrak{L}(N^-)})^{\mathfrak{L}^+(T)}$$

equals the projection

$$\mathcal{C}^{\mathfrak{L}^+(T)} \to (\mathcal{C}_{\mathfrak{L}(N^-)})^{\mathfrak{L}^+(T)},$$

since the averaging with respect to I_u^- does not alter the projection to $\mathfrak{L}(N^-)$-coinvariants.

□

2.4. **The setting for Lin Chen's theorem.**

2.4.1. One way to state Lin Chen's theorem is:

Theorem 2.4.2. *The equivalence* (2.3) *holds for* $\mathcal{C} := \text{D-mod}(\text{Gr}_G)$.

However, this formulation does not reveal the full strength of Lin Chen's theorem for the following reason: one can actually deduce Theorem 2.4.2 from Theorem 2.3.6. This is due to the fact that the action of $\mathfrak{L}^+(T)$ on the categoies

$$\text{D-mod}(\text{Gr}_G)^{\mathfrak{L}(N)} \text{ and } \text{D-mod}(\text{Gr}_G)_{\mathfrak{L}(N^-)}$$

is unipotent-monodromic, which means that they can be explicitly expressed via

$$(\text{D-mod}(\text{Gr}_G)^{\mathfrak{L}(N)})^{\mathfrak{L}^+(T)} \text{ and } (\text{D-mod}(\text{Gr}_G)_{\mathfrak{L}(N^-)})^{\mathfrak{L}^+(T)},$$

respectively.

The real point of Theorem 2.4.2 is that it holds *factorizably*, a statement that cannot be proved by appealing to the Iwahori subgroup. In addition, an analog of Theorem 2.4.2 holds for any parabolic, and the same proof as one indicated below applies.

2.4.3. Here is a reformulation of Theorem 2.4.2, adapted to the form in which we will prove it:

Theorem 2.4.4. *The functor*

$$\text{D-mod}(\text{Gr}_G)^{\mathfrak{L}(N)} \otimes \text{D-mod}(\text{Gr}_G)^{\mathfrak{L}(N^-)} \to \text{D-mod}(\text{Gr}_G) \otimes \text{D-mod}(\text{Gr}_G) \to \text{Vect} \tag{2.4}$$

defines a counit for the duality between $\text{D-mod}(\text{Gr}_G)^{\mathfrak{L}(N)}$ *and* $\text{D-mod}(\text{Gr}_G)^{\mathfrak{L}(N^-)}$.

In (2.4), the arrow

$$\text{D-mod}(\text{Gr}_G) \otimes \text{D-mod}(\text{Gr}_G) \to \text{Vect}$$

is the counit of the canonical self-duality for the category of D-modules on a (ind)=scheme, given by

$$\mathcal{F}_1, \mathcal{F}_2 \mapsto \Gamma_{\text{dR}}(\text{Gr}_G, \mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2).$$

The corresponding equivalence

$$(\text{D-mod}(\text{Gr}_G)^c)^{\text{op}} \to \text{D-mod}(\text{Gr}_G)^c$$

is given by Verdier duality.

2.4.5. Lin Chen's approach to Theorem 2.4.4 consists of explicitly constructing the unit

$$\text{Unit} \in \text{D-mod}(\text{Gr}_G)^{\mathfrak{L}(N)} \otimes \text{D-mod}(\text{Gr}_G)^{\mathfrak{L}(N^-)}$$

and showing that it satisfies the axioms for duality against the counit given by (2.4).

The construction of the unit carries a strong analogy to the construction of the counit in [BK] for the (i^-, r) adjunction, mentioned in Sect. 1.3.8. We outline it below.

2.5. **The Unit as nearby cyces.**

2.5.1. Consider the (ind)-scheme
$$\mathbb{A}^1 \times \mathrm{Gr}_G \times \mathrm{Gr}_G.$$

Let $\gamma : \mathbb{G}_m \to G$ be as in Sect. 1.3.5. Using
$$\mathbb{G}_m \to T \to G \subset \mathfrak{L}^+(G)$$
and the $\mathfrak{L}^+(G)$-action on Gr_G, we obtain a $\mathbb{G}_m$-action on Gr_G.

Consider the graph of this action
$$\mathbb{G}_m \times \mathrm{Gr}_G \overset{\text{graph-of-action}}{\hookrightarrow} \mathbb{G}_m \times \mathrm{Gr}_G \times \mathrm{Gr}_G,$$
and consider
$$\text{graph-of-action}_*(\omega_{\mathbb{G}_m \times \mathrm{Gr}_G}) \in \text{D-mod}(\mathbb{G}_m \times \mathrm{Gr}_G \times \mathrm{Gr}_G).$$

We set
$$\mathrm{Unit} := \Psi(\text{graph-of-action}_*(\omega_{\mathbb{G}_m \times \mathrm{Gr}_G})) \in \text{D-mod}(\mathrm{Gr}_G \times \mathrm{Gr}_G) \simeq \text{D-mod}(\mathrm{Gr}_G) \otimes \text{D-mod}(\mathrm{Gr}_G),$$
where
$$\Psi : \text{D-mod}(\mathbb{G}_m \times \mathrm{Gr}_G \times \mathrm{Gr}_G) \to \text{D-mod}(\mathrm{Gr}_G \times \mathrm{Gr}_G)$$
is the nearby cycles functor.

2.5.2. The above object Unit has interesting stalks. Let Λ denote the coweight lattice of G. For each $\lambda \in \Lambda$, let
$$\iota^\lambda : S^\lambda \hookrightarrow \mathrm{Gr}_G \text{ and } \iota^{-,\lambda} : S^{-,\lambda} \hookrightarrow \mathrm{Gr}_G$$
be the inclusion of the corresponding $\mathfrak{L}(N)$ (resp., $\mathfrak{L}(N^-)$) orbit.

We have
$$(\iota^\lambda \times \iota^{-,\mu})^!(\mathrm{Unit}) \simeq \omega_{S^\lambda \times S^{-\mu}} \otimes \widetilde{\Omega}(\lambda, \mu),$$
where
$$\widetilde{\Omega}(\lambda, \mu) := \mathcal{H}om_{\text{D-mod}(\mathrm{Gr}_G)}(\iota^\mu_!(\omega_{S^\mu}), \iota^\lambda_!(\omega_{S^\lambda}).$$

We can also write
$$(\iota^\mu)^! \circ \iota^\lambda_!(\omega_{S^\lambda}) \simeq \omega_{S^\mu} \otimes \widetilde{\Omega}(\lambda, \mu),$$
so the complexes $\widetilde{\Omega}(\lambda, \mu)$ are closely related to the semi-infinite Kazhdan-Lusztig polynomials.

3. Idea of proof

3.1. **What do we need to show?**

3.1.1. First, we need to show that our object Unit indeed belongs to the subcategory
$$\text{D-mod}(\mathrm{Gr}_G \times \mathrm{Gr}_G)^{\mathfrak{L}(N) \times \mathfrak{L}(N^-)} \subset \text{D-mod}(\mathrm{Gr}_G \times \mathrm{Gr}_G).$$

This follows from the fact that the entire picture is equivariant with respect to the group-indscheme
$$\mathfrak{L}(\widetilde{G}^\gamma),$$
where $\widetilde{G}^\gamma$ is as in Sect. 1.3.7.

3.1.2. In order to prove Theorem 2.4.4, we need to verify that Unit satisfies the duality axioms against (2.4). This amounts to showing that for $\mathcal{F} \in \text{D-mod}(\mathrm{Gr}_G)^{\mathfrak{L}(N)}$, we have
$$(\mathrm{id} \times p)_* \circ (\mathrm{id} \times \Delta)^!(\mathrm{Unit} \boxtimes \mathcal{F}) \simeq \mathcal{F},$$
where $p : \mathrm{Gr}_G \to \mathrm{pt}$.

3.1.3. Consider the following diagram

$$\begin{tikzcd}
\mathbb{G}_m \times \mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathrm{Gr}_G \arrow[r, "j"] & \mathbb{A}^1 \times \mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathrm{Gr}_G & \{0\} \times \mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathrm{Gr}_G \arrow[l, "i"'] \\
\mathbb{G}_m \times \mathrm{Gr}_G \times \mathrm{Gr}_G \arrow[u, "\mathrm{id} \times \Delta"] \arrow[r, "j"] \arrow[d, "\mathrm{id} \times p"'] & \mathbb{A}^1 \times \mathrm{Gr}_G \times \mathrm{Gr}_G \arrow[u, "\mathrm{id} \times \Delta"] \arrow[d, "\mathrm{id} \times p"'] & \{0\} \times \mathrm{Gr}_G \times \mathrm{Gr}_G \arrow[l, "i"'] \arrow[u, "\mathrm{id} \times \Delta"'] \arrow[d, "\mathrm{id} \times p"] \\
\mathbb{G}_m \times \mathrm{Gr}_G \arrow[r, "j"] & \mathbb{A}^1 \times \mathrm{Gr}_G & \{0\} \times \mathrm{Gr}_G. \arrow[l, "i"']
\end{tikzcd} \tag{3.1}$$

We need to show that the (naturally defined) map

$$\Psi \circ (\mathrm{id} \times p)_* \circ (\mathrm{id} \times \Delta)^! \, (\text{graph-of-action}_* (\omega_{\mathbb{G}_m \times \mathrm{Gr}_G}) \boxtimes \mathcal{F}) \to \\ \to (\mathrm{id} \times p)_* \circ (\mathrm{id} \times \Delta)^! \circ \Psi \, (\text{graph-of-action}_* (\omega_{\mathbb{G}_m \times \mathrm{Gr}_G}) \boxtimes \mathcal{F}) \tag{3.2}$$

is an isomorphism, where we note that the LHS in (3.2) is isomorphic to $\mathcal{F}$, and the RHS to

$$(\mathrm{id} \times p)_* \circ (\mathrm{id} \times \Delta)^! (\mathrm{Unit} \boxtimes \mathcal{F}).$$

3.1.4. Thus, we need to understand how to manipulate the nearby cycles sheaf against the operation of !-pullback This is not straightforward, as the nearby cycles do not normally commute with !-pullback.

The diagram (3.1) also involves *-pushforward, but this poses no problem as the morphism in question is proper.

Our ability to control it in the given situation is based on the combination of the several geometric observations.

3.2. Replacing nearby cycles by $i^* \circ j_*$.

3.2.1. First, we claim that we can replace the full nearby cycles functor Ψ in the definition of Unit by unipotent nearby cycles Ψ^{un}. This is a general claim within the following paradigm.

Let us be given a nearby cycles situation

$$\begin{tikzcd}
Y \underset{\mathbb{A}^1}{\times} \mathbb{G}_m \arrow[r, "j"] \arrow[d] & Y \arrow[d] & Y \underset{\mathbb{A}^1}{\times} \{0\} \arrow[l, "i"'] \arrow[d] \\
\mathbb{G}_m \arrow[r] & \mathbb{A}^1 & \{0\}. \arrow[l]
\end{tikzcd} \tag{3.3}$$

Let us assume that we have a $\mathbb{G}_m$-action on Y compatible with standard action of $\mathbb{G}_m$ on $\mathbb{A}^1$. Let

$$\mathcal{F} \in \text{D-mod}(Y \underset{\mathbb{A}^1}{\times} \mathbb{G}_m)$$

be $\mathbb{G}_m$-monodromic.

Lemma 3.2.2.

(a) *The object* $\Psi(\mathcal{F}) \in \text{D-mod}(Y \underset{\mathbb{A}^1}{\times} \{0\})$ *is* $\mathbb{G}_m$*-monodromic.*

(b) *If* $\Psi(\mathcal{F})$ *is* unipotent-monodromic*, then* $\Psi(\mathcal{F}) = \Psi^{\mathrm{un}}(\mathcal{F})$.

The condition of the lemma holds in our case because all objects of $\text{D-mod}(\mathrm{Gr}_G)^{\mathfrak{L}(N)}$ are unipotent-monodromic for the action of T.

Remark 3.2.3. The reason we could not just forget Ψ and use Ψ^{un} instead is that we want to know that Unit has he factorization property.

We can control factorization because the functor Ψ is a priori compatible with external tensor products, while Ψ^{un} is not.

3.2.4. Second, we claim that in a general nearby cycles situation of (3.3), we have

$$\Psi^{\mathrm{un}}(\mathcal{F}) \simeq i^* \circ j_*(\mathcal{F}) \underset{\mathrm{C}^\cdot(\mathbb{G}_m,k)}{\otimes} k.$$

3.2.5. The above two observations have reduced the manipulation of Ψ against !-pullbacks to that of the more manageable functor $i^* \circ j_*$.

There is no problem controlling j_* as it is compatible with !-pullbacks and *-pushforwards. The fact that we can control i^* ultimately comes from Braden's theorem.

3.3. **Using Braden's theorem.**

3.3.1. Let us be given a morphism of schemes over $\mathbb{A}^1$

$$f : Y_2 \to Y_1.$$

Consider the corresponding diagram

$$\begin{tikzcd} Y^1 & Y_1 \underset{\mathbb{A}^1}{\times} \{0\} \arrow[l, "i_1"'] \\ Y^2 \arrow[u, "f"] & Y_2 \underset{\mathbb{A}^1}{\times} \{0\}. \arrow[l, "i_2"'] \arrow[u, "f_0"'] \end{tikzcd}$$

For $\mathcal{F} \in \text{D-mod}(Y_1)$ we have a canonically defined map

$$i_2^* \circ f^!(\mathcal{F}) \to f_0^! \circ i_1^*(\mathcal{F}). \tag{3.4}$$

We want to give sufficient conditions for this map to be an isomorphism.

3.3.2. Suppose that Y_1 and Y_2 carry an action of $\mathbb{G}_m$, in a way compatible with f and the projection of both to $\mathbb{A}^1$. Let

$$Y_i^0 \overset{\mathsf{s}_i^+,\mathsf{q}_i^+}{\leftrightarrows} Y_i^+ \overset{\mathsf{p}_i^+}{\to} Y_i \text{ and } Y_i^0 \overset{\mathsf{s}_i^-,\mathsf{q}_i^-}{\leftrightarrows} Y_i^- \overset{\mathsf{p}_i^-}{\to} Y_i$$

be the corresponding fixed/attractor/repeller loci.

Proposition 3.3.3 (Lin Chen)**.** *Assume that $\mathcal{F}$ is $\mathbb{G}_m$-monodromic. Assume also that:*

(i) *Both sides in* (3.4) *land in a full subcategory of* $\text{D-mod}(Y_2 \underset{\mathbb{A}^1}{\times} \{0\})$ *on which the functor*

$$(\mathfrak{s}_2^-)^! \circ (\mathsf{p}_2^-)^* : \text{D-mod}(Y_2 \underset{\mathbb{A}^1}{\times} \{0\}) \to \text{D-mod}(Y_2^0 \underset{\mathbb{A}^1}{\times} \{0\})$$

is conservative.

(ii) *The diagram*

$$\begin{tikzcd} Y_1^+ \underset{\mathbb{A}^1}{\times} \{0\} \arrow[r, "\mathsf{q}_1^+"] & Y_1^0 \underset{\mathbb{A}^1}{\times} \{0\} \\ Y_2^+ \underset{\mathbb{A}^1}{\times} \{0\} \arrow[u, "f_0^+"] \arrow[r, "\mathsf{q}_2^+"] & Y_2^0 \underset{\mathbb{A}^1}{\times} \{0\} \arrow[u, "f_0^0"'] \end{tikzcd}$$

is Cartesian.

Then the map (3.4) *is an isomorphism.*

3.3.4. We apply this proposition to the upper portion of the diagram (3.1) and the $\mathbb{G}_m$ action on the upper row given by

$$s \cdot (t, g_1, g_2, g_3) = (s \cdot t, s^{-1} \cdot g_1, g_2, g_3),$$

where the last term corresponds to the action of $\mathbb{G}_m$ on Gr_G via γ.

Note that the fixed/repeller locus for this action identifies with

$$\{0\} \times (\mathrm{Gr}_G)^0 \times \mathrm{Gr}_G \times \mathrm{Gr}_G \subset \{0\} \times (\mathrm{Gr}_G)^+ \times \mathrm{Gr}_G \times \mathrm{Gr}_G,$$

where $(\mathrm{Gr}_G)^0 \subset (\mathrm{Gr}_G)^+$ is the fixed/attractor locus of $\mathbb{G}_m$ on Gr_G.

We also note that

$$(\mathrm{Gr}_G)^+ = \underset{\lambda}{\sqcup} S^\lambda, \quad \lambda \in \Lambda,$$

and

$$(\mathrm{Gr}_G)^0 \simeq \mathrm{Gr}_T.$$

References

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