Results and conjectures about parabolic induction for the general linear group over a p-adic field

In honor of Joseph Bernstein's 75th birthday

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Based on joint work with Alberto Mínguez

Fix a local, non-archimedean field F. We will consider finite length complex, smooth representations of the group $\mathrm{GL}_n(F)$, $n \geq 0$. As customary, we denote normalized parabolic induction by $\pi_1 \times \pi_2$.
Our goal is to study irreducibility of $\pi_1 \times \cdots \times \pi_k$.

Theorem (Hernandez, 2010; M. Gurevich–Mínguez)

$\pi_1 \times \cdots \times \pi_k$ is irreducible if and only if $\pi_i \times \pi_j$ is irreducible for all $i < j$.

So we only need to consider the case $k = 2$.

Intertwining operators

Consider the standard intertwining operator

$$M_{\pi_1,\pi_2}(s) : \pi_1 \left|\det\right|^s \times \pi_2 \to \pi_2 \times \pi_1 \left|\det\right|^s$$

given by

$$M_{\pi_1,\pi_2}(\varphi, s)(g) = \int_{\mathrm{Mat}_{n_2\times n_1}(F)} \varphi_s(\begin{pmatrix} & I_{n_1} \\ I_{n_2} & X \end{pmatrix} g) \, dX.$$

The integral converges for $\operatorname{Re} s \gg 0$ and admits (upon realizing $\pi_1 \left|\det\right|^s \times \pi_2$ on the same vector space, independently of s) a meromorphic continuation (as an operator-valued function) to a rational function in q^{-s}. Let

$$\boxed{R_{\pi_1,\pi_2} : \pi_1 \times \pi_2 \to \pi_2 \times \pi_1}$$

be the leading term in the Laurent series at $s = 0$ of $M_{\pi_1,\pi_2}(s)$. It is a non-zero intertwining operator.

Theorem (Seok-Jin Kang, Masaki Kashiwara, Myungho Kim, Se-Jin Oh, 2015)

The following conditions are equivalent for $\pi \in \mathcal{R}(\mathrm{GL}_n)$.

- $\pi \times \pi$ *is irreducible.*
- $\mathrm{End}_{\mathrm{GL}_{2n}(F)}(\pi \times \pi) = \mathbb{C}$.
- $R_{\pi,\pi} : \pi \times \pi \to \pi \times \pi$ *is a scalar.*

In this case we say that π is □-irreducible.

Remark

The theorem (and much more) was proved in a somewhat different context. However, the argument carries over to representations of $\mathrm{GL}_n(F)$.

If π_1, π_2 are □-irreducible and $\pi_1 \times \pi_2$ is irreducible, then $\pi_1 \times \pi_2$ is also □-irreducible. In particular, if π is □-irreducible, then $\pi \times \cdots \times \pi$ is irreducible for any number of factors.

Basic classical results

Theorem

- *(Olshanski, 1974) Every irreducible cuspidal representation (and in fact, every irreducible essentially square-integrable representation) is □-irreducible. (This also works for F non-commutative.)*
- *(Bernstein–Zelevinsky, 1977) Let ρ_1, ρ_2 be irreducible cuspidal representations. Then $\rho_1 \times \rho_2$ is reducible $\iff$ $\rho_2 = \rho_1 \cdot |\det|^{\pm 1}$.*
- *(Zelevinsky, 1980) Every irreducible generic or unramified representation is □-irreducible.*
- *(Bernstein, 1982) Every unitarizable irreducible representation is □-irreducible.*

Unfortunately, contrary to previous expectations, not every irreducible representation is □-irreducible (Leclerc, 2003).

A useful property of $\square$-irreducible representations

Theorem (K[3]O)

Suppose that π is $\square$-irreducible and let $\sigma \in \mathrm{Irr}$. Then,

- $\mathrm{soc}(\pi \times \sigma)$ *and* $\mathrm{soc}(\sigma \times \pi)$ *are irreducible.*
- *Both socles occur with multiplicity one in* $\mathrm{JH}(\pi \times \sigma)$*.*
- $\mathrm{soc}(\pi \times \sigma) = \cos(\sigma \times \pi) = \mathrm{Im}\, R_{\sigma,\pi}$
- $\mathrm{soc}(\sigma \times \pi) = \cos(\pi \times \sigma) = \mathrm{Im}\, R_{\pi,\sigma}$.

Moreover, the following conditions are equivalent.

- $\pi \times \sigma$ *is irreducible.*
- $\sigma \times \pi$ *is irreducible.*
- $R_{\pi,\sigma}$ *is an isomorphism.*
- $R_{\sigma,\pi}$ *is an isomorphism.*
- $\mathrm{soc}(\pi \times \sigma) = \mathrm{soc}(\sigma \times \pi)$.

Zelevinsky classification

For simplicity, from now on we only consider representations that are generated by their fixed points under the Iwahori subgroup, and moreover, their exponents are integers. This entails no loss of generality.

A segment is a set of the form

$$[a, b] = \{n \in \mathbb{Z} : a \leq n \leq b\}$$

for some integers $a \leq b$. To each segment $\Delta = [a, b]$ we attach the one-dimensional character $Z(\Delta) := |\det|^{\frac{a+b}{2}}$ of $\mathrm{GL}_{b-a+1}(F)$. Thus,

$$Z(\Delta) = \mathrm{soc}(|\cdot|^a \times \cdots \times |\cdot|^b).$$

We say that two segments Δ and Δ' are linked if $Z(\Delta) \times Z(\Delta')$ is reducible (of length 2). Equivalently: $\Delta \cup \Delta'$ is a segment but $\Delta \not\subseteq \Delta'$ and $\Delta' \not\subseteq \Delta$.

It is convenient to write $\Delta = [b(\Delta), e(\Delta)]$,
$\overrightarrow{\Delta} = [b(\Delta)+1, e(\Delta)+1]$, $\overleftarrow{\Delta} = [b(\Delta)-1, e(\Delta)-1]$.
In Δ and Δ' are linked then either $e(\overrightarrow{\Delta}) \in \Delta'$ or $e(\overrightarrow{\Delta'}) \in \Delta$, in which case we write $\Delta \prec \Delta'$ or $\Delta' \prec \Delta$ respectively. Thus,

$$\begin{aligned} \Delta \prec \Delta' &\iff b(\Delta) < b(\Delta'), e(\Delta) < e(\Delta') \text{ and } b(\Delta') \leq e(\overrightarrow{\Delta}) \\ \overleftarrow{\Delta} \prec \Delta' &\iff b(\Delta) \leq b(\Delta') \leq e(\Delta) \leq e(\Delta') \\ &\iff b(\Delta') \in \Delta \text{ and } e(\Delta) \in \Delta'. \end{aligned}$$

Given a multisegment $\mathfrak{m} = \Delta_1 + \cdots + \Delta_r$ (a formal sum of segments) we may assume that $\Delta_i \not\prec \Delta_j$ for all $i < j$. Then $\zeta(\mathfrak{m}) := Z(\Delta_1) \times \cdots \times Z(\Delta_r)$ is well defined.

Theorem (Zelevinsky, 1980)

The map

$$\boxed{\mathfrak{m} \mapsto Z(\mathfrak{m}) := \mathrm{soc}(\zeta(\mathfrak{m}))}$$

is a bijection between multisegments and $\mathrm{Irr} = \cup_{n\geq 0} \mathrm{Irr}\, \mathrm{GL}_n(F)$.

Remark

Multisegments classify isomorphism classes of pairs (V, A) consisting of a $\mathbb{Z}$-graded finite-dimensional vector space $V = \oplus_{n\in\mathbb{Z}} V_n$ over $\mathbb{C}$ and a (nilpotent) degree 1 linear map $A : V \to V$ (i.e., $A(V_n) \subset V_{n+1}$ for all $n \in \mathbb{Z}$).
A segment $[a, b]$ corresponds to a (graded) Jordan block ($\dim V_n = 1$ if $n \in [a, b]$ and $V_n = 0$ otherwise; $\operatorname{Ker} A = V_b$).

A basic property is that for any two multisegments $\mathfrak{m}$ and $\mathfrak{m}'$

$$\boxed{Z(\mathfrak{m} + \mathfrak{m}') \text{ occurs with multiplicity one in } \mathrm{JH}(Z(\mathfrak{m}) \times Z(\mathfrak{m}')).}$$

In particular, if $Z(\mathfrak{m}) \times Z(\mathfrak{m}')$ is irreducible, then it is equal to $Z(\mathfrak{m} + \mathfrak{m}')$. This happens if and only if

$$Z(\mathfrak{m}+\mathfrak{m}') = \operatorname{soc}(Z(\mathfrak{m})\times Z(\mathfrak{m}')) \text{ and } Z(\mathfrak{m}+\mathfrak{m}') = \operatorname{soc}(Z(\mathfrak{m}')\times Z(\mathfrak{m})).$$

If at least one of $Z(\mathfrak{m})$ or $Z(\mathfrak{m}')$ is □-irreducible, this simplifies to

$$\boxed{Z(\mathfrak{m} + \mathfrak{m}') \hookrightarrow Z(\mathfrak{m}) \times Z(\mathfrak{m}') \text{ and } Z(\mathfrak{m} + \mathfrak{m}') \hookrightarrow Z(\mathfrak{m}') \times Z(\mathfrak{m}).}$$

A non-□-irreducible representation

The first such example was given by Leclerc (2003). Let

$$\sigma = Z([2,3]+[1,2]) = \mathrm{soc}(|\det|^{\frac{5}{2}}_{\mathrm{GL}_2} \times |\det|^{\frac{3}{2}}_{\mathrm{GL}_2}) \in \mathrm{Irr}\,\mathrm{GL}_4$$

Then $\Sigma := \sigma\,|\det|^2 \times \sigma$ is a representation of GL_8 of length 3:

$$\boxed{\mathrm{JH}(\Sigma) = \mathrm{soc}(\Sigma) + \pi + \cos(\Sigma)}$$

$$\mathrm{soc}(\Sigma) = Z([4,5]+[3,4]+[2,3]+[1,2]),$$

$$\cos(\Sigma) = Z([2,5]+[1,4]),$$

$$\boxed{\pi = Z(\mathfrak{m}),\ \ \mathfrak{m} = [4,5]+[2,4]+[3,3]+[1,2].}$$

Then $\boxed{\pi \times \pi = Z(\mathfrak{m}+\mathfrak{m}) \oplus \mathrm{soc}(\Sigma) \times \cos(\Sigma)}$ is semisimple of length 2.

Classification of regular □-irreducible representations

Theorem (•+Mínguez, 2018)

For any permutation σ of $\{1,\ldots,r\}$, $r \geq 1$ consider the multisegment

$$\mathfrak{m}_\sigma = \sum_{i=1}^{r} [\sigma(i), 2r - i].$$

Then

$$\boxed{Z(\mathfrak{m}_\sigma) \text{ is } \square\text{-irreducible} \iff \sigma \text{ is } 4231 \text{ and } 3412 \text{ avoiding.}}$$

The latter means that there do not exist indices $i < j < k < l$ s.t. either $\sigma(i) > \sigma(k) > \sigma(j) > \sigma(l)$ or $\sigma(k) < \sigma(l) < \sigma(i) < \sigma(j)$.

Remarkably, the same combinatorial condition characterizes smoothness of Schubert varieties of type A (Lakshmibai-Sandhya, 1990)!

Schubert varieties

Consider the complete flag variety

$$B_r(\mathbb{C})\backslash \mathrm{GL}_r(\mathbb{C})$$

consisting of all flags

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_r = \mathbb{C}^r.$$

The group B_r acts with finitely many orbits, indexed by the symmetric group S_r

$$B_r(\mathbb{C})\backslash \mathrm{GL}_r(\mathbb{C}) = \bigcup_{\sigma \in S_r} Y_\sigma.$$

Each orbit $Y_\sigma = B_r \sigma B_r$ is called a Schubert cell. Its closure $X_\sigma = \overline{Y_\sigma}$ is called Schubert variety.
We write $\sigma' \leq \sigma$ if $Y_{\sigma'} \subset X_\sigma$ (Bruhat–Chevalley order). Thus,

$$X_\sigma = \cup_{\sigma' \leq \sigma} Y_{\sigma'}.$$

A multisegment $\mathfrak{m} = [a_1, b_1] + \cdots + [a_r, b_r]$ is called regular if $\#\{a_1, \ldots, a_r\} = \#\{b_1, \ldots, b_r\} = r$. (We do not exclude $a_i = b_j$ for some i, j.)
To a regular multisegment we can attach two permutations $\sigma, \sigma_0 \in S_r$ as follows.
σ encodes the relative order of the a_i's with respect to the b_i's. Explicitly, if we assume (without loss of generality) that $b_1 > \cdots > b_r$ then $a_{\sigma(1)} < \cdots < a_{\sigma(r)}$.
σ_0 encodes how the sets $\{a_1, \ldots, a_r\}$ and $\{b_1, \ldots, b_r\}$ are interwoven. Formally, for all i let $x_i = \#\{j : a_j \leq b_i + 1\}$. Then

$$\sigma_0^{-1}(i) = \max\{j \leq x_i : j \notin \sigma_0^{-1}(\{i+1, \ldots, r\})\}.$$

For instance, if $\max a_i \leq \min b_i + 1$ then $\sigma_0 = \mathrm{id}$.
In general, σ_0 is a stack-sortable permutation in the sense of Knuth, i.e., $\nexists i < j < k$ such that $\sigma_0(j) < \sigma_0(i) < \sigma_0(k)$.
The number of possible σ_0's is the Catalan number of r.
Moreover, $\sigma_0 \leq \sigma$ with an equality if and only if $Z(\mathfrak{m})$ is unramified.

A refinement

Theorem 1 (•+Mínguez)

The following conditions are equivalent for a regular multisegment $\mathfrak{m} = [a_1, b_1] + \cdots + [a_r, b_r]$ with σ and σ_0 as above.

1. $Z(\mathfrak{m})$ *is* $\square$*-irreducible.*
2. *The smooth locus of* X_σ *contains* Y_{σ_0} $(\iff P_{\sigma_0,\sigma} = 1 \iff \#\{t \text{ transposition} : \sigma_0 t \leq \sigma\} = \ell(\sigma))$.
3. *In the Grothendieck group,* $Z(\mathfrak{m})$ *is equal to*

$$\sum_{\sigma' \in S_r : \sigma_0 \leq \sigma' \leq \sigma} \operatorname{sgn} \sigma\sigma' \; Z([a_{\sigma(1)}, b_{\sigma'(1)}]) \times \cdots \times Z([a_{\sigma(r)}, b_{\sigma'(r)}]).$$

Remark In Leclerc's original example:
$\mathfrak{m} = [4,5] + [2,4] + [3,3] + [1,2]$, $\sigma = (4231)$, $\sigma_0 = (1243)$, $P_{\sigma_0,\sigma} = 1 + q$.

More geometry

Fix $V = \oplus_{n\in\mathbb{Z}} V_n$, a finite-dimensional $\mathbb{Z}$-graded $\mathbb{C}$-vector space. Consider the spaces $E_\pm(V)$ of $\mathbb{C}$-linear (nilpotent) endomorphisms of V of degree ± 1, i.e. such that $A(V_n) \subset V_{n\pm 1}$ for all n.
The spaces $E_\pm(V)$ are in duality with respect to the $\mathrm{Aut}(V) = \prod_{n\in\mathbb{Z}} \mathrm{GL}(V_n)$-invariant pairing $A, B \mapsto \mathrm{tr}\, AB = \mathrm{tr}\, BA$.
$\mathrm{Aut}(V)$ acts with finitely many orbits on each of the spaces $E_\pm(V)$.
The orbits are indexed by multisegments (with support determined by $\mathrm{grdim}\, V$). Consider the algebraic set with $\mathrm{Aut}(V)$-action

$$\boxed{\mathfrak{X}(V) = \{(A, B) \in E_+(V) \times E_-(V) : AB = BA\}.}$$

Let $p_\pm : \mathfrak{X}(V) \to E_\pm(V)$ be the projections. (Fibers are vector spaces.)

Theorem (Pyasetskii, 1975)

Each of the two maps $\mathcal{O} \mapsto \overline{p_\pm^{-1}(\mathcal{O})}$ *defines a bijection*

$$\{\mathrm{Aut}(V)\text{-orbits in } E_\pm(V)\} \longleftrightarrow \{\text{irreducible components of } \mathfrak{X}(V)\}.$$

Zelevinsky involution

Let $\mathcal{O}_{\mathfrak{m}}^{\pm}$ be the $\mathrm{Aut}(V)$-orbit in $E_{\pm}(V)$ corresponding to a multisegment $\mathfrak{m}$, $(\mathfrak{C}_{\mathfrak{m}}^{\pm})^{\circ} = p_{\pm}^{-1}(\mathcal{O}_{\mathfrak{m}}^{\pm})$ and $\mathfrak{C}_{\mathfrak{m}}^{\pm} = \overline{(\mathfrak{C}_{\mathfrak{m}}^{\pm})^{\circ}}$. We get a bijection of multisegments $\mathfrak{m} \mapsto \mathfrak{m}^{\#}$ defined by

$$\mathfrak{C}_{\mathfrak{m}}^{+} = \mathfrak{C}_{\mathfrak{m}^{\#}}^{-}$$

The bijection $\mathfrak{m} \mapsto \mathfrak{m}^{\#}$ is an involution which can be described combinatorially by the Moeglin–Waldspurger algorithm (1986). Representation theoretically

$$Z(\mathfrak{m}) = L(\mathfrak{m}^{\#})$$

where $L(\mathfrak{m})$ is the Langlands parametrization of irreducible representations (where $L([a,b])$ is the twist of the Steinberg representation of $\mathrm{GL}_{b-a+1}(F)$ by $|\det|^{\frac{a+b}{2}}$).
From now on we will simply write

$$\mathfrak{C}_{\mathfrak{m}}^{\circ} = (\mathfrak{C}_{\mathfrak{m}}^{+})^{\circ}, \quad \mathfrak{C}_{\mathfrak{m}} = \mathfrak{C}_{\mathfrak{m}}^{+}.$$

A geometric condition of Geiß–Leclerc–Schröer

The following conjecture is a strong form of a special case of a conjecture of Geiß–Leclerc–Schröer (2005).

Conjecture 1

Let $\mathfrak{C}_{\mathfrak{m}}$ be the irreducible component in $\mathfrak{X}(V)$ corresponding to a multisegment $\mathfrak{m}$. Then $Z(\mathfrak{m})$ is $\square$-irreducible if and only if $\mathfrak{C}_{\mathfrak{m}}$ contains an open (i.e., dense) $\mathrm{Aut}(V)$-orbit.

This geometric condition, which we denote by $GLS(\mathfrak{m})$, can be checked very efficiently on a computer.

Relation between Conjecture 1 and Theorem 1

In the course of the proof of Theorem 1 we also proved Conjecture 1 in the case that $\mathfrak{m}$ is regular.
This led Anton Mellit to make the following "duality conjecture".

Conjecture 2 (Anton Mellit)

Let $x, w \in S_n$ with $x \leq w$. Assume that $X_{w_0 x}$ is smooth. Then, the following two conditions are equivalent.

1. *The smooth locus of X_w contains Y_x.*
2. *B_n acts with a dense orbit on the conormal bundle of $Y_{w_0 w}$ in $T^*(X_{w_0 x})$.*

If x is stack-sortable (in which case $X_{w_0 x}$ is smooth), then Theorem 1 and the above imply Conjecture 2 (by an indirect argument).
Similar open vs. smooth duality phenomena appear in other contexts (Fresse-Melnikov, Ben Zvi–Sakellaridis–Venkatesh).

Explication of condition $GLS(\mathfrak{m})$

Fix $A \in \mathcal{O}_{\mathfrak{m}}$ and let G_A be its stabilizer in $\mathrm{Aut}(V)$. Clearly, $GLS(\mathfrak{m})$ is equivalent to the existence of an open G_A-orbit in $p_+^{-1}(A)$, i.e., in the centralizer of A in $E_-(V)$. This is a linear action and we can test this condition by passing to the Lie algebra. If $\mathfrak{m} = \sum_{i \in I} \Delta_i$, then bases for C_A and $\mathrm{Lie}(G_A)$ are indexed by the sets $X_{\mathfrak{m}} = \{(i,j) : \Delta_i \prec \Delta_j\}$ and $Y_{\mathfrak{m}} = \{(i,j) : \overleftarrow{\Delta}_i \prec \Delta_j\}$ resp. Consider the $\mathbb{C}$-vector space $\mathbb{C}^{Y_{\mathfrak{m}}}$ with basis $y_{i,j}$, $(i,j) \in Y_{\mathfrak{m}}$. Then,

Lemma

$GLS(\mathfrak{m})$ holds if and only if there exist $\lambda_{i,j} \in \mathbb{C}$, $(i,j) \in X_{\mathfrak{m}}$, such that the vectors

$$\sum_{k \in I : (k,j) \in X_{\mathfrak{m}}, (i,k) \in Y_{\mathfrak{m}}} \lambda_{k,j} y_{i,k} - \sum_{l \in I : (l,j) \in Y_{\mathfrak{m}}, (i,l) \in X_{\mathfrak{m}}} \lambda_{i,l} y_{l,j}, \quad (i,j) \in X_{\mathfrak{m}}$$

are linearly independent in $\mathbb{C}^{Y_{\mathfrak{m}}}$.

An irreducibility conjecture

Suppose that $\mathfrak{m}$ and $\mathfrak{m}'$ are multisegments, V and V' the corresponding graded vector spaces, $\mathfrak{X}(V)$, $\mathfrak{X}(V')$ the commuting varieties, $\mathfrak{C}_{\mathfrak{m}}$, $\mathfrak{C}_{\mathfrak{m}'}$ the irreducible components of $\mathfrak{X}(V)$, $\mathfrak{X}(V')$. Let $\mathfrak{n} = \mathfrak{m} + \mathfrak{m}'$ and $\mathfrak{C}_{\mathfrak{n}}$ the corresponding irreducible component in $\mathfrak{X}(V \oplus V')$. We have a diagonal embedding

$$\mathfrak{X}(V) \times \mathfrak{X}(V') \subset \mathfrak{X}(V \oplus V')$$

under which

$$\mathfrak{C}_{\mathfrak{m}} \times \mathfrak{C}_{\mathfrak{m}'} \subset \mathfrak{C}_{\mathfrak{n}}.$$

Conjecture 3 (•+Mínguez, 1911.04281)

Suppose that at least one of $Z(\mathfrak{m})$ and $Z(\mathfrak{m}')$ is □-irreducible. Then, $Z(\mathfrak{m}) \times Z(\mathfrak{m}')$ is irreducible if and only if

$$\mathrm{Aut}(V \oplus V') \cdot (\mathfrak{C}_{\mathfrak{m}} \times \mathfrak{C}_{\mathfrak{m}'}) \text{ is dense in } \mathfrak{C}_{\mathfrak{n}}.$$

We will give some evidence for Conjecture 3 below.

Remark

In the case $\mathfrak{m}' = \mathfrak{m}$, the geometric condition is strictly weaker than GLS$(\mathfrak{m})$. Hence Conjecture 1 is not subsumed by (and does not imply) Conjecture 3.
In other words, the assumption that at least one of $Z(\mathfrak{m})$ and $Z(\mathfrak{m}')$ is □-irreducible is not redundant.
In general, the best we could possibly hope for is that

$$\text{“}Z(\mathfrak{m}+\mathfrak{m}') \text{ is a direct summand of } Z(\mathfrak{m}) \times Z(\mathfrak{m}')$$
$$\iff \operatorname{Aut}(V \oplus V') \cdot (\mathfrak{C}_{\mathfrak{m}} \times \mathfrak{C}_{\mathfrak{m}'}) \text{ is dense in } \mathfrak{C}_{\mathfrak{n}}\text{”} \quad (???)$$

It would be constructive to come up with a reasonable conjecture which covers both Conjectures 1 and 3.

Recall that $\mathfrak{C}_{\mathfrak{m}} = \overline{\mathfrak{C}^\circ_{\mathfrak{m}}}$. Let $\mathfrak{Y}$ be the subvariety of $\mathfrak{C}_{\mathfrak{m}+\mathfrak{m}'}$ consisting of the pairs (A, B) satisfying the following two conditions.

1. $A(V'), B(V') \subset V'$, and $(A\big|_{V'}, B\big|_{V'}) \in \mathfrak{C}^\circ_{\mathfrak{m}'}$.
2. The induced pair on the quotient V belongs to $\mathfrak{C}^\circ_{\mathfrak{m}}$.

Conjecture 4

Suppose that at least one of $Z(\mathfrak{m})$ and $Z(\mathfrak{m}')$ is □-irreducible. Then,

$$Z(\mathfrak{m}+\mathfrak{m}') \hookrightarrow Z(\mathfrak{m})\times Z(\mathfrak{m}') \iff \mathrm{Aut}(V\oplus V')\cdot\mathfrak{Y} \text{ is dense in } \mathfrak{C}_{\mathfrak{m}+\mathfrak{m}'}.$$

Conjecture 4 implies Conjecture 3 since under the assumption in green, $Z(\mathfrak{m}) \times Z(\mathfrak{m}')$ is irreducible if and only if $Z(\mathfrak{m}+\mathfrak{m}') \hookrightarrow Z(\mathfrak{m}) \times Z(\mathfrak{m}')$ and $Z(\mathfrak{m}+\mathfrak{m}') \hookrightarrow Z(\mathfrak{m}') \times Z(\mathfrak{m})$.

For all we know, it is possible (?) that Conjecture 4 holds without the assumption in green.

Reformulation

Once again, we can linearize the condition and explicate it as follows. Let $\mathfrak{m} = \sum_{i \in I} \Delta_i$, $\mathfrak{m}' = \sum_{i' \in I'} \Delta_{i'}$. Define

$$X_{\mathfrak{m},\mathfrak{m}'} = \{(i,i') \in I \times I' : \Delta_i \prec \Delta_{i'}\} \text{ (so that } X_{\mathfrak{m}} = X_{\mathfrak{m},\mathfrak{m}})$$

$$Y_{\mathfrak{m},\mathfrak{m}'} = \{(i,i') \in I \times I' : \Delta_i \prec \overrightarrow{\Delta}_{i'}\} \text{ (so that } Y_{\mathfrak{m}} = Y_{\mathfrak{m},\mathfrak{m}})$$

Conjecture 4 (Reformulated)

Assume that at least one of $Z(\mathfrak{m})$ and $Z(\mathfrak{m}')$ is □-irreducible. Then, $Z(\mathfrak{m}+\mathfrak{m}') \hookrightarrow Z(\mathfrak{m}) \times Z(\mathfrak{m}')$ if and only if there exist $\lambda_{i,j}$, $(i,j) \in X_{\mathfrak{m}}$ and $\lambda'_{i,j}$, $(i,j) \in X_{\mathfrak{m}'}$, such that the vectors

$$\sum_{\substack{r \in I : (i,r) \in X_{\mathfrak{m}}, \\ (r,j) \in Y_{\mathfrak{m},\mathfrak{m}'}}} \lambda_{i,r} y_{r,j} - \sum_{\substack{s \in I' : (s,j) \in X_{\mathfrak{m}'}, \\ (i,s) \in Y_{\mathfrak{m},\mathfrak{m}'}}} \lambda'_{s,j} y_{i,s}, \quad (i,j) \in X_{\mathfrak{m},\mathfrak{m}'}$$

are linearly independent in $\mathbb{C}^{Y_{\mathfrak{m},\mathfrak{m}'}}$ (with basis $y_{i,j}$, $(i,j) \in Y_{\mathfrak{m},\mathfrak{m}'}$).

Evidence

Given $a_1 > \cdots > a_k$ and $b_1 > \cdots > b_k$ such that $a_i \le b_i$ we say that the multisegment $\sum_{i=1}^{k}[a_i, b_i]$ (and the corresponding representation) is a ladder.

This is a particularly nice class of representations.

Every ladder representation is □-irreducible.

Theorem (• + Mínguez)

Suppose that $Z(\mathfrak{m})$ is a product of ladder representations. Then, conjectures 3 and 4 hold for any $\mathfrak{m}'$. In particular, this is the case if $Z(\mathfrak{m})$ is a generic, unramified, or unitarizable representation.

Remark

There is a simple combinatorial criterion for the irreducibility of $Z(\mathfrak{m}_1) \times Z(\mathfrak{m}_2)$ where $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are ladders.

Empirical results, 1911.04270

In 1998, Arakawa–Suzuki defined functors from category $\mathcal{O}(\mathfrak{gl}_r)$ to finite-dimensional modules of graded affine Hecke algebras of type A. Using this, the multiplicities of irreducible representations in a standard module $\zeta(\mathfrak{m})$ pertaining to a multisegment $\mathfrak{m} = \Delta_1 + \cdots + \Delta_r$ are given by Kazhdan–Lusztig polynomials with respect to S_r in the same way Verma modules decompose in category $\mathcal{O}(\mathfrak{gl}_r)$.
Given $\mathfrak{m}' = \Delta'_1 + \cdots + \Delta'_{r'}$, we can compute the decomposition of $Z(\mathfrak{m}) \times Z(\mathfrak{m}')$ in the Grothendieck group using Kazhdan-Lusztig polynomials with respect to $S_{r+r'}$.
I computed $P_{x,w}$ for all $x, w \in S_{12}$.
This was feasible since although the order of S_{12} is $\approx 0.5 \times 10^9$, the number of "reduced pairs" in S_{12} is "only" $\approx 46 \times 10^9$. Also, there are "only" $\approx 4.3 \times 10^9$ distinct polynomials of average degree ≈ 10. So we "only" need ≈ 500 GB RAM and we fortunately have a one terabyte RAM machine in our faculty. It took almost a month of CPU time on a single core (December 2017).

With this, I computed the decomposition of $Z(\mathfrak{m}) \times Z(\mathfrak{m})$ in the Grothendieck group for all $\mathfrak{m}$'s comprising at most 6 segments. Following a question by David Kazhdan, I checked the correlation between the length of $Z(\mathfrak{m}) \times Z(\mathfrak{m})$ and the minimal codimension of an $\mathrm{Aut}(V)$-orbit in $\mathcal{C}_{\mathfrak{m}}$ (i.e., the dimension of $\mathcal{C}_{\mathfrak{m}}/\,\mathrm{Aut}(V)$). The results are summarized in the following table. (The first row affirms Conjecture 1 in the cases at hand.)

$\dim \mathcal{C}_{\mathfrak{m}}/\,\mathrm{Aut}(V)$	possible lengths
0	1
1	2
2	6,7,8,9
3	36,37,39,40,44,46,48,53
4	251,257

Moreover, in all cases with length > 2, $Z(\mathfrak{m}) \times Z(\mathfrak{m})$ is not multiplicity-free. (Highest multiplicity is 9.)

A follow-up question by Kazhdan:
Can we detect the length of $Z(\mathfrak{m}) \times Z(\mathfrak{m})$ from other invariants of $\mathcal{C}_{\mathfrak{m}}/\operatorname{Aut}(V)$?

I should point out that I do not know how to refute the condition

$$Z(\mathfrak{m}+\mathfrak{m}) \hookrightarrow Z(\mathfrak{m}) \times Z(\mathfrak{m})$$

(or even the semisimplicity of $Z(\mathfrak{m}) \times Z(\mathfrak{m})$) in any single example. (I do not see any reason why this condition should always hold.)

Decomposition of $\pi_1 \times \pi_2$ in the ladder case

Theorem (Max Gurevich, 2017)

Suppose that π_1 and π_2 are ladder representations. Then $\pi_1 \times \pi_2$ is multiplicity free and $\mathrm{JH}(\pi_1 \times \pi_2)$ admits a simple combinatorial description.

For instance, if

$$\pi_1 = Z(\sum_{i \in \{1,\dots,n\} : i \text{ even}} [i, n+i]), \ \pi_2 = Z(\sum_{i \in \{1,\dots,n\} : i \text{ odd}} [i, n+i])$$

then in the Grothendieck group we have

$$\pi_1 \times \pi_2 = \sum_{\sigma \in S_n : \sigma \text{ 321-avoiding}} Z(\sum_{i=1}^{n} [\sigma(i), i+n])$$

which is of length $C_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}$ (Catalan number). This is the maximum possible length for n segments in total.