A Paley-Wiener theorem for spherical p-adic spaces (and Bernstein morphisms)

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G - conn. red. grp. / local non-arch. F

$$\mathcal{M}(G)^{fl} \subset \mathcal{M}(G) \xrightarrow{\text{Forg}} \text{Vect}$$

$$I : C_c^\infty(G) \to \text{End}\left(\text{Forg}|_{\mathcal{M}(G)^{fl}}\right)$$

$$d \mapsto \left(V \to V : v \mapsto \int_G d(g)\, gv\right)$$

Thm (Bernstein-Heiermann)

- I is injective.
- $T_\cdot \in \text{Im}(I)$ i.f.f.:
  1) $T_\cdot$ is fixed by some ocs in G on both sides.
  2) $P \subset G$ par. E - irr. cusp. rep. of L_P

     χ - unr. ch. of $L_P \rightsquigarrow V_\chi := \text{pind}_P^G(E \otimes \mathbb{C}_\chi)$

     standard to realize all V_χ on single space V

     $v \in V$, $\alpha \in V^\vee$ $\quad \chi \mapsto \alpha(T_{V_\chi}(v))$ should be algebraic

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$\mathcal{A}(G)$ - span of sm. matrix coef. of sm. rep. of f.l.

$$\iota : C_c^\infty(G) \to \mathcal{A}(G)^*$$

$$d \mapsto \left(f \mapsto \int_G f \cdot d\right)$$

Thm

- ι is injective
- $\ell \in \text{Im}(\iota)$ i.f.f.:
  1) ...
  2) $m_\chi(g) = \alpha(g *_\chi v)$ $\qquad \chi \mapsto \ell(m_\chi)$ should be algebraic.

$$X = H \backslash G \qquad \mathcal{A}(X) \subset {}^{sm}C^\infty(X) = \operatorname{colim}_K {}^K C^\infty(X)$$

func. gen. G-rep. of f.l.

$$\iota_X : C_c^\infty(X) \to \mathcal{A}(X)^*$$

$$d \mapsto \left(f \mapsto \int d f\right)$$

$d \longmapsto \left(\int_X f\eta\, \omega \right)$

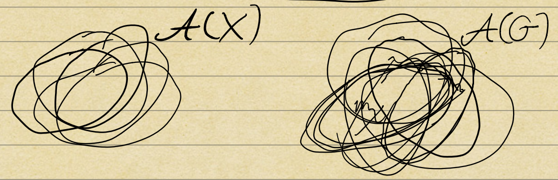
Assume: 1) X is F-spherical

min. par. $P \subset G \Rightarrow P$ has open orbit on X

2) X has fin. mult. prop.

$\forall\ \chi : H \to \mathbb{C}^\times$ sm. $V \in \mathcal{M}(G)^{f\ell} \Rightarrow V / \langle hv - \chi(h) v \rangle$ f.d.

DHS



Thm
- ι_X is injective
- $\ell \in \operatorname{Im}(\iota_X)$ i.f.f.:
  1) ...
  2) ℓ is algebraic

$({}^{sm}C^\infty(X))^{*,cont}$

$\mathcal{A}(X) \subset {}^{sm}C^\infty(X) = \operatorname{colim}_K {}^K C^\infty(X)$

$\lim_K {}^K C_c^\infty(X)$

Thm: bijection ι_X

$\mathcal{A}(X)^{*,alg} \subset \mathcal{A}(X)^*$

V LTVS $\quad V^{*,cont} \subset V^{*,alg}$

Z - aff. $\mathbb{C}$-var.

Z-family in V:

linear map $PD(Z) \xrightarrow{\Phi} V$ s.t.:

functionals on $\mathcal{O}_Z$ which factor via f.d. quot. alg.

1) $\operatorname{Im}(\Phi)$ pre-linearly compact

2) $\forall \ell \in V^{*,cont}$:

$\ell \circ \Phi \in PD(Z)^*$ should be algebraic

$\mathcal{O}_Z \subset PD(Z)^*$

$V := \mathcal{A}(X)$

1) $\operatorname{Im}(\Phi) \subset {}^K\mathcal{A}(X)$ for some cs $K \subset G$

2) $\ell = ev_x : x \in X$

$\ell \in V^*$ is algebraic if $\forall$ Z-family $PD(Z) \xrightarrow{\Phi} V$, $\ell \circ \Phi \in PD(Z)^*$ is algebraic.

$V \subset W$ is <u>PW</u> if

$W^{*,\text{cont.}} \xrightarrow{\sim} V^{*,\text{alg}}$

$\boxed{\begin{array}{c}\underline{\text{Thm}}\\ \mathcal{A}(X) \subset {}^{sm}C^{\alpha}(X)\\ \text{is } \underline{PW}\end{array}}$

$\mathcal{M}$ – $\mathbb{C}$-lin. ab. cat. closed under colimits

$\mathcal{M}^{fl} \subset \mathcal{M}$

$D, C \in \mathrm{Fun}^{\ell}(\mathcal{M}, \mathrm{Vect})$

$$\begin{tikzcd}
 & \mathrm{Hom}(D, C) \arrow[dl] \arrow[d] \\
\mathrm{Hom}^{alg}(D|_{\mathcal{M}^{fl}}, C|_{\mathcal{M}^{fl}}) \subset & \mathrm{Hom}(D|_{\mathcal{M}^{fl}}, C|_{\mathcal{M}^{fl}})
\end{tikzcd}$$

$t : D|_{\mathcal{M}^{fl}} \to C|_{\mathcal{M}^{fl}}$ is <u>algebraic</u> if:

$\forall$ Z-aff. $\mathbb{C}$-var., $M \in \mathcal{M}$, $\mathcal{O}_Z \to \mathrm{End}(M)$, $\quad W \subset Z$ 0-dim. cl. $\Downarrow$ $M|_W \in \mathcal{M}^{fl}$

$d \in D(M)$, $\alpha : C(M) \to \mathcal{O}_Z$ $\mathcal{O}_Z$-lin.

$\left(d \in D(M) \xrightarrow{t} C(M) \xrightarrow{\alpha} \mathcal{O}_Z \right)$

$\exists\, a \in \mathcal{O}_Z$ s.t.: $\forall\, W \subset Z$ 0-dim. cl.

$$\begin{tikzcd}
d \in D(M) \arrow[d] \arrow[ddrrr, bend right] & & & \mathcal{O}_Z \ni a \arrow[d] & a \arrow[d, mapsto] \\
D(M|_W) \arrow[r, "t"] & C(M|_W) \simeq C(M)|_W \arrow[rr, "\alpha"] & & \mathcal{O}_W \ni \square & \square \\
 & & & \heartsuit &
\end{tikzcd}$$

require $\heartsuit = \square$

<u>Prop:</u> Under some conditions on $\mathcal{M}$ and C, for any D:

$$\begin{array}{c}\mathrm{Hom}(D, C)\\ \downarrow \wr\\ \mathrm{Hom}^{alg}(D|_{\mathcal{M}^{fl}}, C|_{\mathcal{M}^{fl}}).\end{array}$$

Note: $t: D|_{\mathcal{M}^{sc}} \to C|_{\mathcal{M}^{sc}}$ is algebraic if:

- it is algebraic for $Z = \mathbb{A}^1$.
- it is generically algebraic for $Z = \mathbb{A}^n$.

$\mathcal{M} := \mathcal{M}(G)$
$D := \mathrm{Forg}$
$C := \mathrm{Coinv}_H$

$(X = H\backslash G)$

$\lim_K {}^K C_c^\infty(X) \simeq$

$$\begin{array}{ccc} \left({}^{sm}C^\infty(X)\right)^{*,\mathrm{cont}} & \simeq & \mathrm{Hom}(\mathrm{Forg}, \mathrm{Coinv}_H) \\ \downarrow i_X & & \downarrow \text{restriction} \\ \mathcal{A}(X)^{*,\mathrm{alg}} & \simeq & \mathrm{Hom}^{\mathrm{alg}}(\mathrm{Forg}|_{\mathcal{M}(G)^{sc}}, \mathrm{Coinv}_H|_{\mathcal{M}(G)^{sc}}) \end{array}$$

$V \to {}_H V$

$V := C_c^\infty(G) \to C_c^\infty(H\backslash G)$

e_K

$$\begin{array}{c} f \in \mathcal{A}(X) \\ \downarrow ev_{x_0} \\ \mathbb{C} \end{array}$$

$V \otimes V^{*,H} \to \mathbb{C}$

Claim Cond. of Prop. hold in our case

Bernstein's theory

AGS: $\mathfrak{Z}_K \circlearrowright {}^K C_c^\infty(X)$ f.g.

$H\backslash G \rightsquigarrow H_0\backslash G$ $\qquad$ $B: C_c^\infty(H_0\backslash G) \longrightarrow C_c^\infty(H\backslash G)$

$\gamma \in G$ s.t.:

1) γ sits in a split torus.

2) $\forall$ ocs $K \subset G$, $\exists$ $1 \in U \subset G$ open s.t. $n >> 0$:

$$_H[g\gamma^n e_K] = {}_H[\gamma^n e_K]$$

$e_K * \gamma^n * e_K$ $\qquad$ V^K

Ex $G = SL_2$, $H = SO_2$

$\gamma = \begin{pmatrix} \pi^{-1} & 0 \\ 0 & \pi \end{pmatrix}$

def. of H_0

$g \in H_0$ if:

- $\forall$ ocs $K \subset G$, $\forall n >> 0$:

$L \qquad {}_H[Y^n g e_K] = {}_H[Y^n e_K]$

~~$\mathfrak{J}_+(V) \otimes \mathfrak{J}_-(V^\vee) \to \mathbb{C}$~~

Ex $(H \cap T)\cdot U^- \subset H_0$
$\left(\overset{\text{probably}}{=}\right)$

~~$\Delta G \subset G \times G$~~

$U \times U$

$\text{asymp} \in \mathcal{A}(H\backslash G)^{*, H_0}$

$\text{asymp}(f) \qquad \phi : n \mapsto f(H Y^n)$

$\parallel \qquad \phi = \phi_{=0} + \phi_{\neq 0}$

$\phi_{\neq 0}(0)$

Frob. res :

G-inv. $\quad \text{Asymp} : \mathcal{A}(H\backslash G) \to \mathcal{A}(H_0\backslash G)$

Thm Asymp is cont.

asymp $\iff b : \text{Coinv}_{H_0}|_{\mathcal{M}(G)^{fl}} \to \text{Coinv}_H|_{\mathcal{M}(G)^{fl}}$

Thm b is algebraic.

- generically algebraic
- algebraic for $Z = \mathbb{A}^1$:
  - * Bernstein's stabilization lemma

$+ \left[\begin{array}{l} A = \mathbb{C}[y] \quad A \subset M \text{ f.g.} \\ T \in \text{End}_A(M) \text{ invertible} \\ \exists \text{ monic } f \in A[x], \text{ whose free coef. is a unit in } A, \text{ s.t.} \\ f(T) = 0. \end{array} \right]$

$V / \langle e_K Y e_K \rangle^{K} V$