A Paley-Wiener theorem for spherical p-adic spaces and Bernstein morphisms (unpolished notes for a talk)

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The Paley-Wiener theorem in the group case

Let G be a connected reductive group over a non-archimedean local field F.

Let us first remind the Paley-Wiener theorem of Bernstein and Heiermann.

We have a map

$$I : C_c^{\infty}(G) \to \mathrm{End}(\mathrm{Forg}|_{\mathcal{M}(G)^{\mathrm{fl}}})$$

given by

$$d \mapsto \left(V \to V : \ v \mapsto \int_G d(g) \cdot gv \right)_{V \in \mathcal{M}(G)^{\mathrm{fl}}} .$$

Here $\mathcal{M}(G)^{\mathrm{fl}} \subset \mathcal{M}(G)$ is the full subcategory of the category of smooth G-modules consisting of those of finite length, and $\mathrm{Forg} : \mathcal{M}(G) \to \mathrm{Vect}$ is the forgetful functor.

Theorem (The Paley-Wiener theorem, Bernstein and Heiermann)

The map I is injective, and its image consists of those $T_\bullet = (T_V)_{V \in \mathcal{M}(G)^{\text{fl}}}$ satisfying:

1. *$T_\bullet$ is fixed on both sides by some ocs $K \subset G$.*
2. *Given a parabolic $P \subset G$ and a cuspidal irrep E of the Levi L_P, it is standard to realize all $V_\chi := pind_P^G(E \otimes \mathbb{C}_\chi)$, for χ an unramified character of L_P, on the same space V. Given $v \in V$ and $\alpha \in V^\vee$, one demands $\chi \mapsto \alpha(T_{V_\chi} v)$ to be an algebraic function.*

In addition to the above "representation-theoretic" formulation, let us give an "harmonic-analytic" formulation. Let us consider the space $\mathcal{A}(G)$ of functions on G which is the span of smooth matrix coefficients of smooth representations of finite length.

We have a map

$$\iota : C_c^{\infty}(G) \to \mathcal{A}(G)^*$$

given by $\iota(d) = \left(f \mapsto \int_G d \cdot f\right)$. Then we can formulate:

The Paley-Wiener theorem in the group case

Theorem (The Paley-Wiener theorem, harmonic analysis reformulation)

The map ι is injective, and its image consists of functionals $\ell \in \mathcal{A}(G)^*$ satisfying:

1. *ℓ is fixed on both sides by some ocs $K \subset G$.*
2. *In the above notation, we consider $m_\chi(g) := \alpha(g *_\chi v)$ and demand $\chi \mapsto \ell(m_\chi)$ to be an algebraic function.*

Statement of the main theorem

We are now interested in a "relative" version of the above.

Let $X := H\backslash G$ be a homogeneous G-space. We can consider $\mathcal{A}(X)$, the space of smooth functions on X whose G-translates span a smooth G-module of finite length. We have a map

$$\iota_X : C_c^\infty(X) \to \mathcal{A}(X)^*, \quad \phi \mapsto \left(f \mapsto \int_X f \cdot \phi \right).$$

The group case is when $G^{\text{new}} := G \times G$ and $H^{\text{new}} := \Delta G$.

Statement of the main theorem

It turns out that a natural class of the spaces X to restrict attention to at that point is that of F-**spherical** spaces. Those are the X's for which, given a minimal parabolic $P \subset G$, P admits an open orbit on X.

Moreover, we will assume another property of X, that of **finite multiplicity**. This means that, for any $V \in \mathcal{M}(G)^{\text{fl}}$ and any smooth character $\chi : H \to \mathbb{C}^{\times}$, the space of coinvariants

$$V / \langle hv - \chi(h)v \rangle_{h \in H, v \in V}$$

is finite dimensional.

Conjecturally, the second always holds given the first, and this is known in several families of cases (for example by works of Delorme and Sakellaridis-Venkatesh).

Let us notice that Delorme, Harinck and Sakellaridis have a work regarding Paley-Wiener theorems for (some) spherical spaces. At least morally speaking, the above theorem says that we have families m_χ (given by parabolic induction) which define some "algebraic coordinate charts" for $\mathcal{A}(G)$, and a functional is representable if it is algebraic along these charts. At least rhetorically, the work of Delorme, Harinck and Sakellaridis can be said to describe such charts for $\mathcal{A}(X)$ by using Bernstein morphisms along boundary degenerations. The idea of the approach I will give here is to be ignorant about Bernstein morphisms. So how to have charts? One can try to use all possible "algebraic" charts using some definition of algebraicity. But then, it is not clear how to see that there are enough such charts. Again morally speaking, we will be able to see that there are enough charts just by knowledge of the above charts for G - in some sense seeing that our situation is "covered well" by that of G thanks to the sphericity.

Statement of the main theorem

Theorem (Our main theorem - A Paley-Wiener type theorem)

Assume that X is F-spherical and satisfies the finite multiplicity property. Then the map ι_X is injective, and its image coincides with the subspace of functionals ℓ such that:

1. *ℓ is fixed by some open compact subgroup $K \subset G$.*
2. *ℓ is* algebraic*.*

We will want next to explain, what are the algebraic functionals mentioned in the theorem.

Algebraic functionals

Let us reformulate the theorem. Consider the space ${}^{sm}C^\infty(X) = \mathrm{colim}_K {}^K C^\infty(X)$ ($\mathcal{A}(X)$ sits inside it). We give it the linear topology of the colimit, where each term has the linear topology defined by requiring point evaluations to be continuous. To $\mathcal{A}(X)$ we give the subspace topology.

Then $({}^{sm}C^\infty(X))^{*,cont} = \lim_K {}^K C_c^\infty(X)$ and the restriction map $({}^{sm}C^\infty(X))^{*,cont} \to \mathcal{A}(X)^*$ is basically our ι_X. So the theorem reformulates to stating that the restriction map

$$({}^{sm}C^\infty(X))^{*,cont} \to \mathcal{A}(X)^{*,alg}$$

is an isomorphism.

Algebraic functionals

Given an affine $\mathbb{C}$-variety Z, we denote by $PD(Z)$ the space of functionals on $\mathcal{O}_Z$ which can be factored via a finite-dimensional quotient algebra. There is an inclusion $\mathcal{O}_Z \hookrightarrow PD(Z)$.

A Z-**family** of functions in $\mathcal{A}(X)$ is a linear map $\Phi : PD(Z) \to \mathcal{A}(X)$ such that there exists an ocs $K \subset G$ for which $\mathrm{Im}(\Phi) \subset {}^K\mathcal{A}(X)$, and such that for every $x \in X$ the functional $PD(Z) \xrightarrow{\Phi} \mathcal{A}(X) \xrightarrow{\mathrm{ev}_x} \mathbb{C}$ belongs to $\mathcal{O}_Z$.

A functional $\ell \in \mathcal{A}(X)^*$ is **algebraic** if, for every Z-family $\Phi : PD(Z) \to \mathcal{A}(X)$ as above, the functional $PD(Z) \xrightarrow{\Phi} \mathcal{A}(X) \xrightarrow{\ell} \mathbb{C}$ belongs to $\mathcal{O}_Z$.

Algebraic functionals

One can try to axiomatize the situation by introducing a concept which has nothing to do with p-adic representation theory, as follows. Given a linearly topologized V, one defines a Z-family $\Phi : PD(Z) \to V$ similarly to the above - the image $\mathrm{Im}(\Phi)$ should be pre-linearly compact, and, for every continuous functional $\ell \in V^*$, the composition $\ell \circ \Phi$ belongs to $\mathcal{O}_Z$. One says that a (not-necessarily continuous) functional $\ell \in V^*$ is **algebraic** if for every Z-family $\Phi : PD(Z) \to V$, the functional $\ell \circ \Phi$ belongs to $\mathcal{O}_Z$. We then say that V is **Paley-Wiener** if algebraic functionals on V are continuous.

A categorical setting - 1

In order to approach the problem by means of representation theory, we work in the following categorical framework.

Let $\mathcal{M}$ be a $\mathbb{C}$-linear abelian category admitting all small colimits, and denote by $\mathcal{M}^{\text{fl}} \subset \mathcal{M}$ the full subcategory consisting of objects of finite length.

Given functors $C, D \in \text{Fun}^{l}(\mathcal{M}, \text{Vect})$, we consider the restriction map

$$\text{Hom}(D, C) \to \text{Hom}(D|_{\mathcal{M}^{\text{fl}}}, C|_{\mathcal{M}^{\text{fl}}}).$$

We can describe a subspace (of **algebraic** morphisms)

$$\text{Hom}^{\text{alg}}(D|_{\mathcal{M}^{\text{fl}}}, C|_{\mathcal{M}^{\text{fl}}}) \subset \text{Hom}(D|_{\mathcal{M}^{\text{fl}}}, C|_{\mathcal{M}^{\text{fl}}})$$

in which clearly the image of the restriction map is contained.

Namely, $t \in \mathrm{Hom}(D|_{\mathcal{M}^{\mathrm{fl}}}, C|_{\mathcal{M}^{\mathrm{fl}}})$ is **algebraic** if given a $\mathbb{C}$-variety Z, $M \in \mathcal{M}$ equipped with a $\mathcal{O}_Z$-action, $d \in D(M)$ and $\alpha \in \mathrm{Hom}_{\mathcal{O}_Z}(M, \mathcal{O}_Z)$, we demand there to exist $f \in \mathcal{O}_Z$ such that for every 0-dim. closed $W \subset Z$ the image of f under $\mathcal{O}_Z \to \mathcal{O}_W$ agrees with the image of d under

$$D(M) \to D(M|_W) \to C(M|_W) \cong C(M)|_W \xrightarrow{\alpha} \mathcal{O}_Z|_W = \mathcal{O}_W.$$

Relating the two

Let us explain how the two frameworks relate. We set $\mathcal{M} := \mathcal{M}(G)$, $D :=$ Forg the forgetful functor and $C := \mathrm{Coinv}_H$ the functor of H-coinvariants

$$\mathrm{Coinv}_H(M) := {}_H M := M/\langle hm - m\rangle_{h\in H, m\in M}.$$

We have the following commutative diagram:

$$\begin{array}{ccc} \lim_K {}^K\mathcal{S}(X) & \xleftrightarrow{\sim} & \mathrm{Hom}(Forg, Coinv_H) \\ \downarrow \iota_X & & \downarrow \text{restriction} \\ \mathcal{A}(X)^* & \xleftrightarrow{\sim} & \mathrm{Hom}(Forg|_{\mathcal{M}(G)^{\mathrm{fl}}}, \mathrm{Coinv}_H|_{\mathcal{M}(G)^{\mathrm{fl}}}) \end{array}.$$

Relating the two

Let us describe the lower isomorphism. Given $\ell \in \mathcal{A}(X)^*$, we need to provide a linear map $V \to {}_H V$ for $V \in \mathcal{M}(G)^{\mathrm{fl}}$. This is the same as a bilinear pairing

$$V \otimes V^{*,H} \to \mathbb{C}.$$

Given $v \in V$ and $\alpha \in V^{*,H}$, we set the value of the pairing to be $\ell(m_{v,\alpha})$ where $m_{v,\alpha}(Hg) := \alpha(gv)$.

A categorical setting - 2

Proposition (Categorical Paley-Wiener type theorem)

Suppose that there are enough compact projective objects P in $\mathcal{M}$ which can be equipped with an $\mathcal{O}_Z$-action (for a $\mathbb{C}$-variety Z) such that:

1. *For any 0-dim. closed $W \subset Z$, $P|_W$ has finite length.*
2. *Moreover, any morphism $M \to V$ to a finite length object we can factor via some $P \to P|_W$.*
3. *$C(P)$ is a finitely generated $\mathcal{O}_Z$-module.*

Then, for any D, restriction sets an isomorphism

$$\mathrm{Hom}(D, C) \xrightarrow{\sim} \mathrm{Hom}^{\mathrm{alg}}(D|_{\mathcal{M}^{\mathrm{fl}}}, C|_{\mathcal{M}^{\mathrm{fl}}}).$$

A categorical setting - 2

Claim

The conditions of the proposition are satisfied for $\mathcal{M} := \mathcal{M}(G)$ and $C := Coinv_H$.

The claim uses Bernstein's theory of decomposition of $\mathcal{M}(G)$ (the projective generators and so on), and a statement of [Aizenbud-Gourevitch-Sayag] saying that the K-invariants of $\mathcal{S}(H\backslash G)$ being a finitely generated $\mathcal{H}_K$-module.

In the setting of the above proposition, we can also formulate conditions for a given

$$t \in \mathrm{Hom}(D|_{\mathcal{M}^{\mathrm{fl}}}, C|_{\mathcal{M}^{\mathrm{fl}}})$$

to belong to $\mathrm{Hom}^{\mathrm{alg}}(D|_{\mathcal{M}^{\mathrm{fl}}}, C|_{\mathcal{M}^{\mathrm{fl}}})$ which require less checking than in the definition of being an algebraic morphism. Namely, it is enough to check that t is algebraic when testing on $Z := \mathbb{A}^1$, and generically algebraic when testing on $Z := \mathbb{A}^n$ for $n > 1$. Here, "generically algebraic" means that the condition holds when we restrict to an open dense $U \subset \mathbb{A}^n$.

Bernstein morphisms

Let us be given an element $\gamma \in G$ which sits in a split torus and satisfies:

- For any ocs $K \subset G$ there exists a neighbourhood of 1 in G such that for large enough n one has ${}_H[g\gamma^n e_K] = {}_H[\gamma^n e_K]$ for g in this neighbourhood.

The "degeneration" of H is then the closed subgroup $H_0 \subset G$ consisting of elements $g \in G$ such that

- For any ocs $K \subset G$, for large enough n one has ${}_H[\gamma^n g e_K] = {}_H[\gamma^n e_K]$.

For example: $G = SL_2, \theta(g) = (g^t)^{-1}, H = G^\theta, \gamma = \begin{pmatrix} \pi^{-1} & 0 \\ 0 & \pi \end{pmatrix}$.
Then I believe that $H_0 = (H \cap T) \cdot U^-$ (certainly the right is contained is in the left, I didn't check the equality formally).

Bernstein morphisms

Basically following Casselman's classical canonical pairing, we have an H_0-invariant functional

$$\text{asymp} \in \mathcal{A}(H\backslash G)^{*,H_0}$$

given as follows. Let $f \in \mathcal{A}(H\backslash G)$. Consider the sequence $\phi : n \mapsto f(H\gamma^n)$. It is a finite vector under the shift operator. Hence We can decompose uniquely $\phi = \phi_0 + \phi_{\neq 0}$ where the shift-invariant subspace spanned by ϕ_0 (resp. $\phi_{\neq 0}$) has only generalized shift-eigenvalues 0 (resp. not equal to 0). We then set $\text{asymp}(\phi) := \phi_{\neq 0}(0)$.

By Frobenius reciprocity, we obtain a G-invariant linear map

$$\text{Asymp} : \mathcal{A}(H\backslash G) \to \mathcal{A}(H_0\backslash G).$$

Bernstein morphisms

Theorem

The map Asymp *is continuous (with respect to the linear topologies discusses previously).*

Corollary

Taking continuous duals, we obtain the Bernstein map

$$B : C_c^\infty(H_0\backslash G) \to C_c^\infty(H\backslash G).$$

To prove the theorem, we again pass to a representation-theoretic reformulation. Namely, the functional asymp corresponds to a morphism

$$b \in \mathrm{Hom}(\mathrm{Forg}|_{\mathcal{M}(G)^{\mathrm{fl}}}, \mathrm{Coinv}_H|_{\mathcal{M}(G)^{\mathrm{fl}}})$$

and equivalent to the above theorem is:

Theorem

The morphism b is algebraic.

To prove this, I use the criterion mentioned above. I check that b is generically algebraic (not very hard). Then it is left to check that b is algebraic on Z-families for $Z := \mathbb{A}^1$. This I check using Bernstein's stabilization lemma and the following fact: Let M be a finitely generated module over $A := \mathbb{C}[y]$ and $T \in \mathrm{End}_A(M)$ invertible. Then there exists a monic polynomial $f \in A[x]$ with invertible free coefficient such that $f(T) = 0$.