Langlands conjecture for curves over local fields

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Introduction

In this talk I present conjectures and results of joint work with Edward Frenkel and Pavel Etingof on a Langlands conjecture for curves over local fields. Namely, we propose an analytic approach to the geometric Langlands correspondence.

The idea to develop such a theory (over $\mathbb{C}$) was suggested recently by R. Langlands, who attempted to construct Hecke operators in this setting. A similar idea and some results in this direction were discussed in my work with A. Braverman (2005) and by M. Kontsevich in his paper "Notes on motives in finite characteristic" (2007) and letters (2019).

My goal is to define Hecke operators over a local field F and formulate a number of conjectures. In particular, in the case of the complex field we characterize eigenfunctions of these operators as eigenstates of the quantum Hitchin system, and their eigenvalues in terms of real opers for the Langlands dual group. This implements some ideas of J. Teschner.

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In formulations of the conjectures I will often assume the validity of previously formulated conjectures. Also I restrict the presentation to the case $G = PGL_2$.

Our conjectures are proved only for curves of small genus g (with parabolic points). We assume that there are at least four parabolic points for $g = 0$ and at least one for $g = 1$.

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Algebro-geometric preliminaries

Let:

- X be a smooth complete curve of genus $g(X)$ over a field F;
- $X_0^2 \subset X^2$ the open subvariety of pairs of distinct points;
- K the canonical bundle on X;
- $K^{1/2}$ a square root of K;
- $\widetilde{\mathrm{Bun}}$ the stack of rank 2 vector bundles on X of degree 1;
- $\mathrm{Bun} \subset \widetilde{\mathrm{Bun}}$ the substack of stable bundles, which is actually a smooth complete variety.

We choose a line bundle $\mathcal{O}(1)$ on X of degree 1, and write $\mathcal{F}(n) := \mathcal{F} \otimes \mathcal{O}(1)^{\otimes n}$ for any vector bundle $\mathcal{F}$ on X and $n \geq 0$.

Let $\widetilde{Z}$ be the Hecke stack whose F-points are tuples $(\mathcal{F}, \mathcal{G}, \alpha, (x, y))$ with

- $\mathcal{F}, \mathcal{G} \in \widetilde{\mathrm{Bun}}$;
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We denote by $\widetilde{p}_1, \widetilde{p}_2 : \widetilde{Z} \to \widetilde{\mathrm{Bun}}$ and $\widetilde{q} : \widetilde{Z} \to X_0^2$ the natural projections, and set $\widetilde{q}_i := \widetilde{p}_i \times \widetilde{q}$. The projections $\widetilde{q}_i$ are smooth fibrations with fibers isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. We denote by Ω_2 the canonical bundle along the fibers of $\widetilde{q}_2$. Let $Z \subset \widetilde{Z}$ be the subvariety of tuples $(\mathcal{F}, \mathcal{G}, \alpha, (x, y))$ such that $\mathcal{F}, \mathcal{G} \in \mathrm{Bun}$, and denote by p_i, q_i, q the restrictions of the morphisms $\widetilde{p}_i, \widetilde{q}_i, \widetilde{q}$ to Z. For $\mathcal{G} \in \mathrm{Bun}, (x, y) \in X_0^2$ we denote by $Z_{\mathcal{G}}^{x,y}$ the fiber of q_2 over $(\mathcal{G}, (x, y))$. Note that the variety $Z_{\mathcal{G}}^{x,y}$ is 2-dimensional and not proper in general.

Beilinson and Drinfeld associated to any choice of $K^{1/2}$ a square root $\Omega^{1/2}$ of the canonical bundle Ω on $\widetilde{\mathrm{Bun}}$. They also constructed an isomorphism

$$a : \widetilde{q}_1^{\star}(\Omega^{1/2}) \to \widetilde{q}_2^{\star}(\Omega^{1/2}) \otimes \Omega_2 \otimes \widetilde{q}^{\star}(K^{-1/2} \boxtimes K^{-1/2}).$$

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We denote by $\widetilde{p}_1, \widetilde{p}_2 : \widetilde{Z} \to \widetilde{\mathrm{Bun}}$ and $\widetilde{q} : \widetilde{Z} \to X_0^2$ the natural projections, and set $\widetilde{q}_i := \widetilde{p}_i \times \widetilde{q}$. The projections $\widetilde{q}_i$ are smooth fibrations with fibers isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. We denote by Ω_2 the canonical bundle along the fibers of $\widetilde{q}_2$. Let $Z \subset \widetilde{Z}$ be the subvariety of tuples $(\mathcal{F}, \mathcal{G}, \alpha, (x, y))$ such that $\mathcal{F}, \mathcal{G} \in \mathrm{Bun}$, and denote by p_i, q_i, q the restrictions of the morphisms $\widetilde{p}_i, \widetilde{q}_i, \widetilde{q}$ to Z. For $\mathcal{G} \in \mathrm{Bun}, (x, y) \in X_0^2$ we denote by $Z_{\mathcal{G}}^{x,y}$ the fiber of q_2 over $(\mathcal{G}, (x, y))$. Note that the variety $Z_{\mathcal{G}}^{x,y}$ is 2-dimensional and not proper in general.

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From now on we assume that F is a local field. Using the norm map $a \mapsto |a|$ from $F^\star$ to $\mathbb{C}^\star$, we associate with a line bundle $\mathcal{L}$ on any smooth algebraic F-variety $\mathbb{Y}$ a $\mathbb{C}$-line bundle $|\mathcal{L}|$ on the analytic F-variety $Y := \mathbb{Y}(F)$ (with structure group $\mathbb{R}_{>0}$). For archimedian fields the bundle $|\mathcal{L}|$ has a C^∞-structure.

Example 1

1. If $\mathcal{L} = \mathcal{O}_{\mathbb{Y}}$ then $|\mathcal{L}|$ is the sheaf of smooth functions on Y (where as usual in the non-archimedian case, smooth means locally constant).
2. If $\mathcal{L}$ is the canonical bundle then $|\mathcal{L}|$ is the sheaf of smooth measures on Y.

Denote by $\mathcal{S}(Y)$ the space of smooth global $\mathbb{C}$-valued sections of $|\Omega_{\mathbb{Y}}|^{1/2}$; i. e., $\mathcal{S}(Y)$ is the space of smooth half-densities on Y. We have a positive Hermitian form on $\mathcal{S}(Y)$, $(f,g) := \int_Y f\overline{g}$, and we let $L^2(Y)$ denote the Hilbert space completion of $\mathcal{S}(Y)$.

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In our case when $\mathbb{Y} = \mathrm{Bun}$, we write $V := \mathcal{S}(\mathrm{Bun})^J$ and $\mathcal{H} := L^2(\mathrm{Bun})^{\mathrm{J}}$, where J is the Jacobian of X. Thus V is the space of half-densities invariant under tensoring with degree 0 line bundles, (or, equivalently, half-densities on the space of PGL_2-bundles.)

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Definition of the Hecke operators, ctd.

Now we may define the **Hecke operators** by

$$(T_{x,y}f)(\mathcal{G}) := \int_{Z_{\mathcal{G}}^{x,y}} p_1^\star(f),\ f \in V.$$

One can show the existence of a dense Zariski open set $\mathrm{Bun}^\circ \subset \mathrm{Bun}$ such that for any $f \in V$ this integral is absolutely convergent for $\mathcal{G} \in \mathrm{Bun}^\circ$ and belongs to the space $\widehat{V}$ of smooth functions on Bun°. Thus we obtain a linear operator

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This defines a map $T : V \times X_0^2 \to \widehat{V}$ by $T(f,x,y)(\mathcal{G}) := (T_{x,y}f)(\mathcal{G})$. Note that neither $|a|$ nor T depend on the choice of $K^{1/2}$.

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The compactness conjecture

Conjecture 2 (Compactness)

1. The operators $T_{x,y} : V \to \widehat{V}$ land in $\mathcal{H}$ and extend to bounded operators on $\mathcal{H}$.
2. The operators $T_{x,y}$ are compact.
3. $\bigcap_{(x,y)\in X_0^2} \mathrm{Ker} T_{x,y} = 0$.

Note that this conjecture implies that the operators $T_{x,y}$ commute and are self-adjoint.

The validity of the first two parts of the Compactness conjecture would follow from the following purely algebraic Rational singularities conjecture.

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2. The operators $T_{x,y}$ are compact.
3. $\bigcap_{(x,y)\in X_0^2} \mathrm{Ker} T_{x,y} = 0$.

Note that this conjecture implies that the operators $T_{x,y}$ commute and are self-adjoint.

The validity of the first two parts of the Compactness conjecture would follow from the following purely algebraic Rational singularities conjecture.

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The validity of the first two parts of the Compactness conjecture would follow from the following purely algebraic Rational singularities conjecture.

Rational singularities

We denote by $r_n : P_n \to \mathrm{Bun} \times \mathrm{Bun}$ the projective bundle whose fibers $P_n^{\mathcal{F},\mathcal{G}}$ are the projective spaces $\mathbb{P}\mathrm{Hom}(\mathcal{F}, \mathcal{G}(n))$ (note that the dimension of this space is independent on $\mathcal{F}, \mathcal{G}$ if n is large enough). Let $P_n^\circ \subset P_n$ be the open set of morphisms $\alpha : \mathcal{F} \to \mathcal{G}$ such that $\det \alpha \not\equiv 0$.

We denote by $X_0^{2n} \subset X^{2n}$ the subvariety of distinct $2n$-tuples of points. For $D = (x_1, \dots, x_n; y_1, \dots, y_n) \in X_0^{2n}$ we denote by $P_n^\circ(D) \subset P_n^\circ$ the closed subvariety of points for which $\mathrm{supp}(\mathrm{Coker}(\alpha)) = D$. For $n \gg 1$ $P_n^\circ(D)$ are smooth varieties containing $Z^{x_1,y_1} \times_{\mathrm{Bun}} Z^{x_2,y_2} \cdots \times_{\mathrm{Bun}} Z^{x_n,y_n}$ as a dense open subset, where $Z^{x,y} = q^{-1}(x, y)$. Denote by $P_n(D)$ the closure of $P_n^\circ(D)$ in P_n.

Conjecture 3 (Rational singularities)

The proper morphisms $P_n(D) \to \mathrm{Bun} \times \mathrm{Bun}$ have rational singularities for $n \gg g(X)$.

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Rational singularities, ctd.

Remark 4

Let $T_D := \prod_{i=1}^{n} T_{x_i,y_i}$. The Rational singularities conjecture implies that T_D extends to a bounded endomorphism of $\mathcal{H}$ defined by a continuous kernel $K_D \in \Gamma(\mathrm{Bun} \times \mathrm{Bun}, |S| \boxtimes |S|)$. So T_D is a trace class operator and

$$\mathrm{Tr}(T_D) = \int_{\mathrm{Bun}} \Delta_*(K_D)$$

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Semistable bundles

Remark 5

One can consider bundles of arbitrary degrees n. It is known that stable rank 2 vector bundles form a complete variety if and only if n is odd. However, equivalence classes of semistable rank 2 vector bundles always form a normal complete variety, albeit not necessarily smooth. Note that $\mathrm{Bun}_1 = \mathrm{Bun}$ and that tensoring with $\mathcal{O}(1)$ defines an isomorphism $\mathrm{Bun}_n \cong \mathrm{Bun}_{n+2}$. Thus we have two different spaces Bun_0 and Bun_1 which give rise to Hilbert spaces $\mathcal{H}(0)$ and $\mathcal{H}(1)$. We can introduce (initially densely defined) Hecke operators T_x on $\mathcal{H}(0) \oplus \mathcal{H}(1)$ (swapping $\mathcal{H}(0)$ and $\mathcal{H}(1)$). We expect that T_x define bounded, compact, commuting self-adjoint operators and $T_{x,y} = T_x T_y$. If so then the theory should be the same as before, except the kernels K_D are expected to have logarithmic discontinuities at semistable bundles (this is what happens in examples).

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From now we assume the validity of the Compactness conjecture. Denote by $\mathcal{T} \subset \mathrm{End}(\mathcal{H})$ the commutative algebra generated by the Hecke operators, and by R the spectrum of $\mathcal{T}$. The Compactness conjecture implies the existence of a direct sum decomposition $\mathcal{H} = \oplus_{r \in R} \mathcal{H}_r$, where $\mathcal{H}_r \subset \mathcal{H}$ are finite-dimensional $\mathcal{T}$-invariant subspaces; i. e., it implies that the spectrum of $\mathcal{T}$ is discrete.

A Langlands conjecture for X would provide a description of the set R. Unfortunately, for non-archimedian fields we do not have any conjectural description of R. So from now on we assume that $F = \mathbb{C}$. We recall that the norm for $\mathbb{C}$ used in the definition of $|\mathcal{L}|$ is the **square** of the usual absolute value, $|z| = \mid z \mid^2$.

For a complex manifold Y denote by $\mathcal{D}(Y) \subset \mathrm{End}(\mathcal{S}(Y))$ the algebra of global (twisted) holomorphic differential operators on Y acting on $\underline{\Omega_Y^{1/2}}$. For $D \in \mathcal{D}(Y)$ we define $\overline{D} \in \mathrm{End}(\mathcal{S}(Y))$ by $\overline{D}(f) := \overline{D(\overline{f})}$ and denote by $\mathcal{A}(Y) \subset \mathrm{End}(\mathcal{S}(Y))$ the subalgebra generated by D and $\overline{D}, D \in \mathcal{D}(Y)$.

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For a complex manifold Y denote by $\mathcal{D}(Y) \subset \mathrm{End}(\mathcal{S}(Y))$ the algebra of global (twisted) holomorphic differential operators on Y acting on $\Omega_Y^{1/2}$. For $D \in \mathcal{D}(Y)$ we define $\overline{D} \in \mathrm{End}(\mathcal{S}(Y))$ by $\overline{D}(f) := \overline{D(\overline{f})}$ and denote by $\mathcal{A}(Y) \subset \mathrm{End}(\mathcal{S}(Y))$ the subalgebra generated by D and $\overline{D}$, $D \in \mathcal{D}(Y)$.

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Let $Y := \mathrm{Bun}$, $\mathcal{D} := \mathcal{D}(\mathrm{Bun})$ and $\mathcal{A} := \mathcal{A}(\mathrm{Bun})$. As shown by Beilinson and Drinfeld, the algebra $\mathcal{D}$ is the image of the center of the completed universal enveloping algebra of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$ at the critical level; therefore, it is commutative.

The following statement is easy to prove.

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The algebra $\mathcal{A}$ commutes with the Hecke operators.

As follows from the Compactness conjecture, for any $r \in R$ we obtain a finite-dimensional representation ρ_r of the algebra $\mathcal{A}$ on $\mathcal{H}_r$.

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The following description of the variety $\mathrm{Spec}(\mathcal{D})$ is due to Beilinson and Drinfeld.

The isomorphism $H^1(X, K) \cong \mathbb{C}$ defines a canonical extension $\mathcal{M}$ of $K^{-1/2}$ by $K^{1/2}$.

Definition 7

1. An SL_2-oper on X is an SL_2-connection ∇ on $\mathcal{M}$. The space of SL_2-opers is denoted by Op.
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The monodromy of an SL_2-connection ∇ on $\mathcal{M}$ defines an equivalence class ρ_∇ of homomorphisms $\pi_1(X) \to SL_2(\mathbb{C})$. The representation ρ_∇ is known to be irreducible and non-unitarizable.

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Claim 8

1. There is a bijection $\nabla \mapsto D_\nabla$ between opers ∇ and formally self-adjoint second order differential operators $D = \partial^2 + u : K^{-1/2} \to K^{3/2}$.
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Eigenvalues of Hecke operators

Conjecture 9 (Simple spectrum)

$\dim(\mathcal{H}_r) = 1$ for all $r \in R$.

If so, then there exists a map $\kappa : R \to \mathrm{Op}$ such that the algebra $\mathcal{A} = \mathcal{D} \otimes \overline{\mathcal{D}}$ acts on $\mathcal{H}_r$ through the homomorphism $(\kappa_r, \overline{\kappa}_r) \in \mathrm{Op}^2$.

Conjecture 10 (Real opers)

1. The map $\kappa : R \to \mathrm{Op}$ is an embedding.
2. $\mathrm{Im}(\kappa) = \mathrm{Op}_{\mathbb{R}}$, the set of real opers.

For $\nabla \in \mathrm{Op}_{\mathbb{R}}$ write $r := \kappa^{-1}(\nabla)$.

Since the operators $T_{x,y}, (x, y) \in X_0^2$ act on the subspace $\mathcal{H}_r \subset \mathcal{H}$ as scalars, the restriction of the Hecke operator T to $\mathcal{H}_r$ defines a section $\phi_\nabla \in \Gamma(X_0^2, K^{-1/2} \boxtimes K^{-1/2})$ (the eigenvalue of T.)

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$\dim(\mathcal{H}_r) = 1$ for all $r \in R$.

If so, then there exists a map $\kappa : R \to \mathrm{Op}$ such that the algebra $\mathcal{A} = \mathcal{D} \otimes \overline{\mathcal{D}}$ acts on $\mathcal{H}_r$ through the homomorphism $(\kappa_r, \overline{\kappa}_r) \in \mathrm{Op}^2$.

Conjecture 10 (Real opers)

1. The map $\kappa : R \to \mathrm{Op}$ is an embedding.
2. $\mathrm{Im}(\kappa) = \mathrm{Op}_{\mathbb{R}}$, the set of real opers.

For $\nabla \in \mathrm{Op}_{\mathbb{R}}$ write $r := \kappa^{-1}(\nabla)$.

Since the operators $T_{x,y}, (x,y) \in X_0^2$ act on the subspace $\mathcal{H}_r \subset \mathcal{H}$ as scalars, the restriction of the Hecke operator T to $\mathcal{H}_r$ defines a section $\phi_\nabla \in \Gamma(X_0^2, K^{-1/2} \boxtimes K^{-1/2})$ (the eigenvalue of T.)

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Results

Conjecture 11 (Eigenvalues)

Up to scaling, $\phi_\nabla = s^x_\nabla \boxtimes s^y_\nabla$.

We have proved all the conjectures for curves of genus 0 and some for curves of genus ≤ 2. Since for curves X of genus ≤ 1 the space of stable rank 2 bundles modulo tensoring with line bundles is either empty or consists of one point, we consider the case of parabolic bundles.

Fix a collection of distinct points $S = \{t_1, \dots, t_k\} \subset X(F)$. An rank 2 S-parabolic bundle is a rank 2 vector bundle $\mathcal{F}$ on X with a choice of one-dimensional subspaces in fibers $\mathcal{F}_t, t \in S$. It is easy to extend the definition of the Hecke operators $T_{x,y}$ for $x, y \notin S$ and the formulation of our conjectures to the parabolic case.

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Examples

Let $X = \mathbb{P}^1$ and $S = \{t_0, \ldots, t_m, \infty\}$, $m \geq 2$. Then $\mathrm{Bun}_0 \cong \mathrm{Bun}_1 \cong (\mathbb{P}^1)^{m+2} /\!\!/ PGL_2 \cong (\mathbb{P}^1)^{m+1} /\!\!/ (z \mapsto az + b)$ (we identify Bun_0 with Bun_1 using the Hecke modification at ∞ along the parabolic line). So an element of $\mathcal{H} = L^2(\mathrm{Bun}_0) = L^2(\mathrm{Bun}_1)$ may be viewed as a function $\psi(y_0, \ldots, y_m)$, $y_i \in F$, invariant under simultaneous shift of all variables and homogeneous of degree $-m/2$ (so it represents a half-density on Bun).

The Hecke operator $T_x : L^2(\mathrm{Bun}_1) \to L^2(\mathrm{Bun}_0)$ is given by

$$(T_x\psi)(y_0, \ldots, y_m) = \int_F \psi\left(\frac{t_0 - x}{s - y_0}, \ldots, \frac{t_m - x}{s - y_m}\right) \frac{|ds|}{\prod_{i=0}^m |s - y_i|}.$$

This operator is self-adjoint and compact, and $\cap_x \mathrm{Ker} T_x = 0$. If $F = \mathbb{R}$ or $\mathbb{C}$ then T_x commutes with the Gaudin hamiltonians, which are the quantum Hitchin hamiltonians in this case.

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where $f(t,x,y,z) := (xy + xz + yz - t)^2 + 4(1 + t - x - y - z)xyz$. and F^2 is the set of squares in F.

In the archimedean case T_x commutes with the Lamé operator

$$L = \partial_z z(z-1)(z-t)\partial_z + z$$

which is the quantum Hitchin hamiltonian in this case, and for $F = \mathbb{C}$ the eigenfunctions of T_x are the single-valued joint eigenfunctions of $L, \overline{L}$. This example has been discussed by Kontsevich.

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Remark

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Let ϕ be a compactly supported $3/2$-density on X_0^{2n}. Then one can consider the operator $T_\phi = \int_{X_0^{2n}(F)} \phi(D)T_D$. To obtain a discrete spectral decomposition of $\mathcal{H}$ under the algebra of Hecke operators, it suffices to prove that T_ϕ is well defined and compact.

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