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GAP SETS FOR SPECTRA OF CUBIC GRAPHS

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JOINT WORK WITH
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SOME OF JOSEPH'S IMPACT ON ME

• 1987 SABBATICAL JERUSALEM

• HIS PROOF OF THE MEROMORPHIC CONTINUATION OF EISENSTEIN SERIES (1980'S TO 2020 !), SOME HISTORICAL COMMENTS.

• HIS IDEA (WITH KAZHDAN) TO GIVE BOUNDS TOWARDS SELBERG'S EIGENVALUE CONJECTURE USING THE DICHOTOMY THAT THE DIMENSIONS OF IRREDUCIBLE REPRESENT-IONS OF CHEVALLEY GROUPS $G(\mathbb{Z}/p\mathbb{Z})$. ARE EITHER ONE DIMENSIONAL OR VERY LARGE.

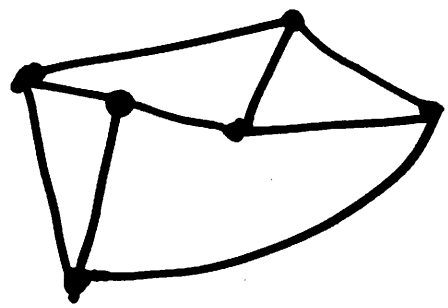
$\Rightarrow$ REALIZED IN XUE-SARNAK AND IS THE "END-GAME" IN PROOFS OF EXPANSION IN THIN MATRIX GROUPS.

• HIS WORKS WITH ANDRE REZNIKOV ON SUB-CONVEX ESTIMATES FOR L-FUNCTIONS AND PERIODS.

⋮

<u>LESSON</u>: IF AND WHEN JOSEPH HAS SOMETHING TO SAY, LISTEN CAREFULY IT IS ALWAYS GOLD.

X: THE SET OF FINITE CONNECTED 3-REGULAR GRAPHS.



THIS ONE IS PLANAR.

- FOR $Y \in X$, $\sigma(Y)$ IS THE SPECTRUM OF THE ADJACENCY MATRIX A_Y "LAPLACIAN"

$$A_Y f(x) = \sum_{y \sim x} f(y) \; ; \; f : V(Y) \to \mathbb{C}$$

SELF-ADJOINT

$$\sigma(Y) \subset [-3, 3]$$

3 IS SIMPLE

-3 IS AN EIGENVALUE IFF Y IS BIPARTITE.

QUESTION: WHAT GAPS CAN BE CREATED IN $\sigma(Y)$ FOR LARGE Y'S.

• <u>CELEBRATED GAP</u> (OPTIMAL EXPANDERS)

ALON-BOPANNA $(2\sqrt{2}, 3)$, IT IS A MAXIMAL INTERVAL AND IS ACHIEVED BY RAMANUJAN GRAPHS.
(LUBOTZKY-PHILLIPS-S, MARGULIS 86/87)
RECENTLY MARCUS-SPIELMAN-SRIVASTAVA USING LEE-YANG + INTERLACING)

• TIGHT BINDING HAMILTONIANS IN PHYSICS ASK FOR A GAP AT -3 (KOLLAR-FITZPATRICK-HOUCK-S) ...

• IN THE CHEMISTRY OF LARGE CARBON CLUSTERS (EG FULLERENES) A GAP AT 0 IS DECISIVE (HUCKELL ORBITAL STABILITY).

GAP AT −3 HOFFMAN GRAPHS

IF Z IS ANY CONNECTED GRAPH

$L(Z)$ ITS LINE GRAPH:

VERTICES OF $L(Z)$ ARE EDGES OF Z
JOIN TWO IF THEY SHARE A VERTEX.

- FACTORIZATION VIA ~~ADJACENCY~~ INDIDENCE MATRIX

$$\sigma(L(Z)) = \{-2\}^{m-n} \cup \sigma(-2I + A_Z + D_Z)$$

$$\subset [-2, \infty)$$

DIAG WITH VALENCE

m = # OF EDGES OF Z
n = # OF VERTICES.

SO $\lambda_{min}(L(Z)) \geq -2$; HOFFMAN GRAPH.

FROM $$\lambda_{min}(Z) = \min_{x \neq 0} \frac{\langle x, A_Z x\rangle}{\langle x, x\rangle}$$

IT FOLLOWS THAT FOR ANY INDUCED SUBGRAPH B OF Z

$$\lambda_{min}(Z) \leq \lambda_{min}(B).$$

SO IF Z IS A HOFFMAN GRAPH THEN IT CANNOT CONTAIN A HOST OF SMALL INDUCED SUBGRAPH.

$\Rightarrow$ CLASSIFICATION OF HOFFMAN GRAPHS

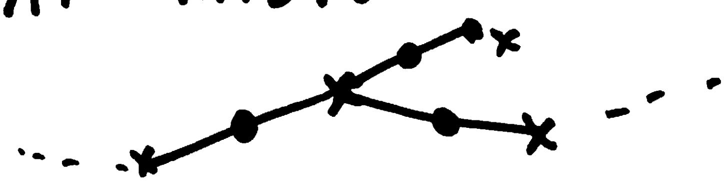
CAMERON - GOETHELS - SEIDEL - SHULT (1975)
"LINE GRAPHS, ROOT SYSTEMS AND ELLIPTIC GEOMETRY"

EXCEPT FOR A FINITE LIST OF SPORADIC GRAPHS THESE ARE ALL GENERALISED LINE GRAPHS.

- TO CONSTRUCT $\overline{\text{LINE GRAPHS}}$ IN X, DEFINE

$$T : X \to X \qquad \text{BY}$$

$y \to S(y)$. SUBDIVIDE y BY ADDING VERTICES AT MIDPOINTS OF EDGES



YIELDS A 2-3 REGULAR GRAPH, THEN

$$T(y) := L(S(y)) \in X.$$

$$|T(y)| = 3|y|.$$

PROPOSITION (K-F-H-S)

IF $y \in X$ AND IS LARGE THEN $\sigma(y) \subset [-2,3]$ IFF $y = T(z)$ FOR SOME $z \in X$.

$\Rightarrow$ $[-3,-2)$ IS A MAXIMAL GAP INTERVAL.

DEFINITION: A CLOSED SUBSET K OF $[-3,3]$ IS A SPECTRAL SET IF THERE ARE INFINITELY MANY y'S IN X SUCH THAT $\sigma(y) \subset K$.

$[-3,3] \setminus K$ IS A GAP SET.

WE SEEK MAXIMAL GAP SETS OR MINIMAL SPECTRAL SETS.

SIMILAR QUESTION IN OTHER SETTINGS:

- FOR QUOTIENTS OF HIGHER RANK SYMMETRIC SPACES S (OR BRUHAT-TITS BUILDINGS)

RIGIDITY RESULTS OF (ABERT, BERGERON BIRINGER, GELANDER, NIKOLOV, RAINBAULT, SAMET) SHOW THAT IF

$$Y_\Gamma = S/\Gamma \quad , \Gamma \text{ A LATTICE IN } \mathrm{ISO}(S)$$

THEN AS $\mathrm{VOL}(Y_\Gamma) \to \infty$, Y_Γ CONVERGES BENYAMINI-SCHRAMM TO S, IN PARTICULAR

$\sigma(Y_\Gamma)$ (OR AT LEAST THE TEMPERED PART)

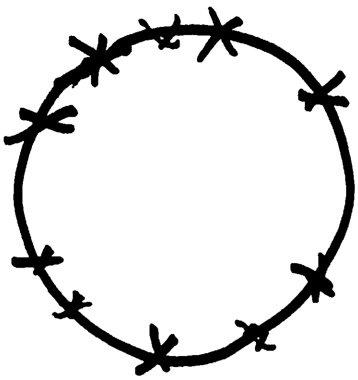
BECOMES DENSE IN THE SUPPORT OF THE PLANCHAREL MEASURE.

SO <u>NO GAPS</u>!

VERY RIGID.

- ZEROS OF ZETA FUNCTIONS OF CURVES AND ABELIAN VARIETIES OVER A FIXED $\mathbb{F}_q$ ($g \to \infty$). (TFASMAN, VLADUT, DRINFELD, SERRE) IN CONNECTION WITH GOPPA CODES

$|\lambda| = \sqrt{q}\ \epsilon$



$2g$ OF THEM SYMMETRIC

WHAT KIND OF GAP SETS $K \subset S'$ CAN BE ACHIEVED.

- FOR CURVES TFASMAN | VLADUT NO GAPS CAN BE CREATED.

- FOR ABELIAN VARIETIES $A/\mathbb{F}_q$ SERRE (2018) SHOWS THAT ESSENTIALLY AS LONG AS K HAS TRANFINITE DIAMETER AT LEAST $q^{1/4}$ THEN IT CAN BE ACHIEVED.

THEOREM FEKETE (1930)

LET $K \subset \mathbb{C}$ BE COMPACT,

IF $\text{CAP}(K) = \text{TRANFINITE DIAMETER}(K) < 1$

THEN

$$\{\alpha : \alpha \text{ ALGEB. INTEGER WITH ALL ITS GALOIS CONJ IN } K\}$$

IS FINITE.

SHARP S^1 ; $\text{CAP}(S^1) = 1$ CONTAINS ROOTS OF 1.

$d(K)$: $$d_n = \max_{i<j} |z_i - z_j|^{2/n(n-1)}, \quad z_j \in K$$

$d_n \searrow d$ = TRANSFINITE DIAMETER.

BACK TO SPECTRA OF CUBIC GRAPHS.

THEOREM 1: (K-S)

EVERY POINT ξ IN $[-3,3)$ IS PLANAR GAPPED, THAT IS THERE IS NBH V_ξ OF ξ SUCH THAT

$$V_\xi \cap \sigma(Y_j) = \phi \quad \text{FOR } Y_j \text{ PLANAR}, \ |Y_j| \to \infty.$$

PROPOSITION 2 (K-S)

IF $K \subset [-3,3]$ IS SPECTRAL THEN

$$\text{CAP}(K) \geqslant 1.$$

SO SPECTRAL SETS CANNOT BE TOO SMALL.

THEOREM 2 EXTREMAL GAP INTERVALS

(i) $(-1,1)$ IS A MAXIMAL GAP INTERVAL ABOUT $\xi=0$ FOR BIPARTITE GRAPHS.

(ii) $(-2,0)$ IS A MAXIMAL SYMMETRIC GAP INTERVAL ABOUT $\xi=-1$.

THE MAP $T : X \to X$ SATISFIES

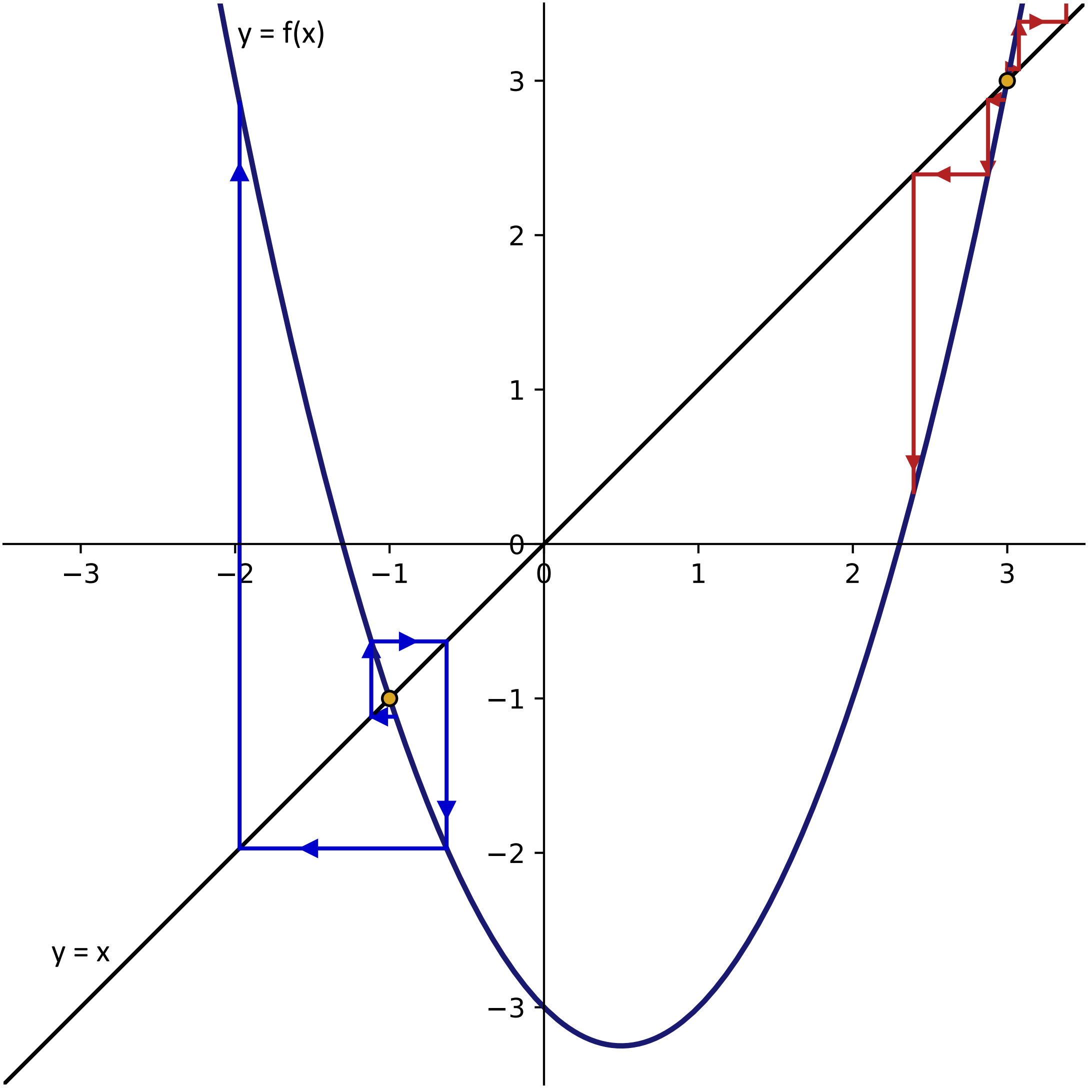
$$\sigma(T(y)) = f^{-1}(\sigma(y)) \cup \{0\}^{n/2} \cup \{-2\}^{n/2}$$

$n = |V(y)|$.

WHERE

$$f(x) = x^2 - x - 3$$

OUR PROOFS MAKE USE OF THE DYNAMICS OF T ON X AND OF f ON $\mathbb{R}$.



y = f(x)
y = x
−3
−2
−1
0
1
2
3
3
2
1
0
−1
−2
−3

$$f^{-1}([-3,3]) = [-2,0] \sqcup [1,3]$$

$$[-3,3] \supset f^{-1}([-3,3]) \supset f^{-2}([-3,3]) \cdots$$

SET

$$\Lambda = \bigcap_{m=0}^{\infty} f^{-m}([-3,3])$$

Λ IS A CANTOR SET.

$f^m(x) \to \infty$ AS $m \to \infty$ IF $x \notin \Lambda$.

$f|_\Lambda$ IS TOPOLOGICALLY EQUIVALENT TO THE SHIFT ON $\{0,1\}^{\mathbb{N}}$.

LET

$$A = \Lambda \cup \bigcup_{m=0}^{\infty} f^{-m}\{(0)\}.$$

A IS CLOSED AND CONSISTS OF THE CANTOR SET Λ AND THE OTHER POINTS ARE ISOLATED AND ACCUMULATE ON Λ.

<u>THEOREM 3:</u> A IS A MINIMAL SPECTRAL SET, $CAP(A) = 1$ AND $\{y \in X : \sigma(y) \subset A\}$ CONSISTS OF FINITELY MANY T-ORBITS.

<u>PROPOSITION:</u>

- IF K IS SPECTRAL THEN SO IS

$$f^{-1}(K) \cup \{0\} \cup \{-2\}$$

AND IF K IS MINIMAL SPECTRAL THEN SO IS $f^{-1}(K) \cup \{0\} \cup \{-2\}$.

- IF $f^k(\xi)$ IS PLANAR GAPPED ~~THEN~~ FOR SOME $k \geq 0$ THEN SO IS ξ.

MAXIMAL GAP INTERVALS I

I	$[-3,-2)$	$(-2,0)$	$(-1,1)$	$(2\sqrt{2},3)$
REALIZED WITH	PLANAR	NON-PLANAR	NON-PLANAR	CANNOT BE PLANAR

MINIMAL SPECTRAL SETS K.

	1	2
K	$[-2\sqrt{2}, 2\sqrt{2}] \cup \{3\}$	A
	CANNOT BE REALIZED WITH PLANAR	PLANAR

$CAP(A) = 1$ ABSOLUTELY MINIMAL

$CAP(K_1) = \sqrt{2}$ PERHAPS THE MAX CAPACITY OF MINIMAL SPECTRAL SETS.

DISCUSSION.

K_1 IS MINIMAL SPECTRAL : ABERT | GLASNER | VIRAG.

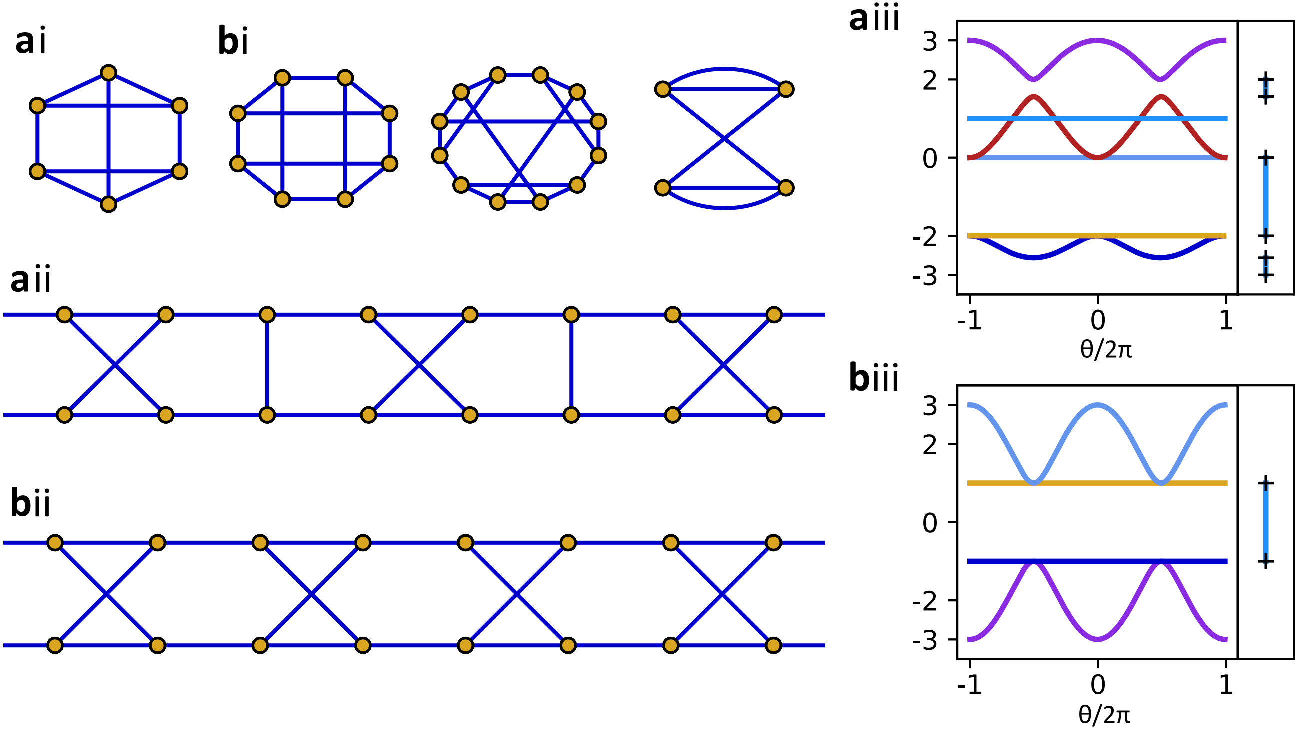
OUTLINE OF PROOFS:

- THE MAP T AND ITS DYNAMICS

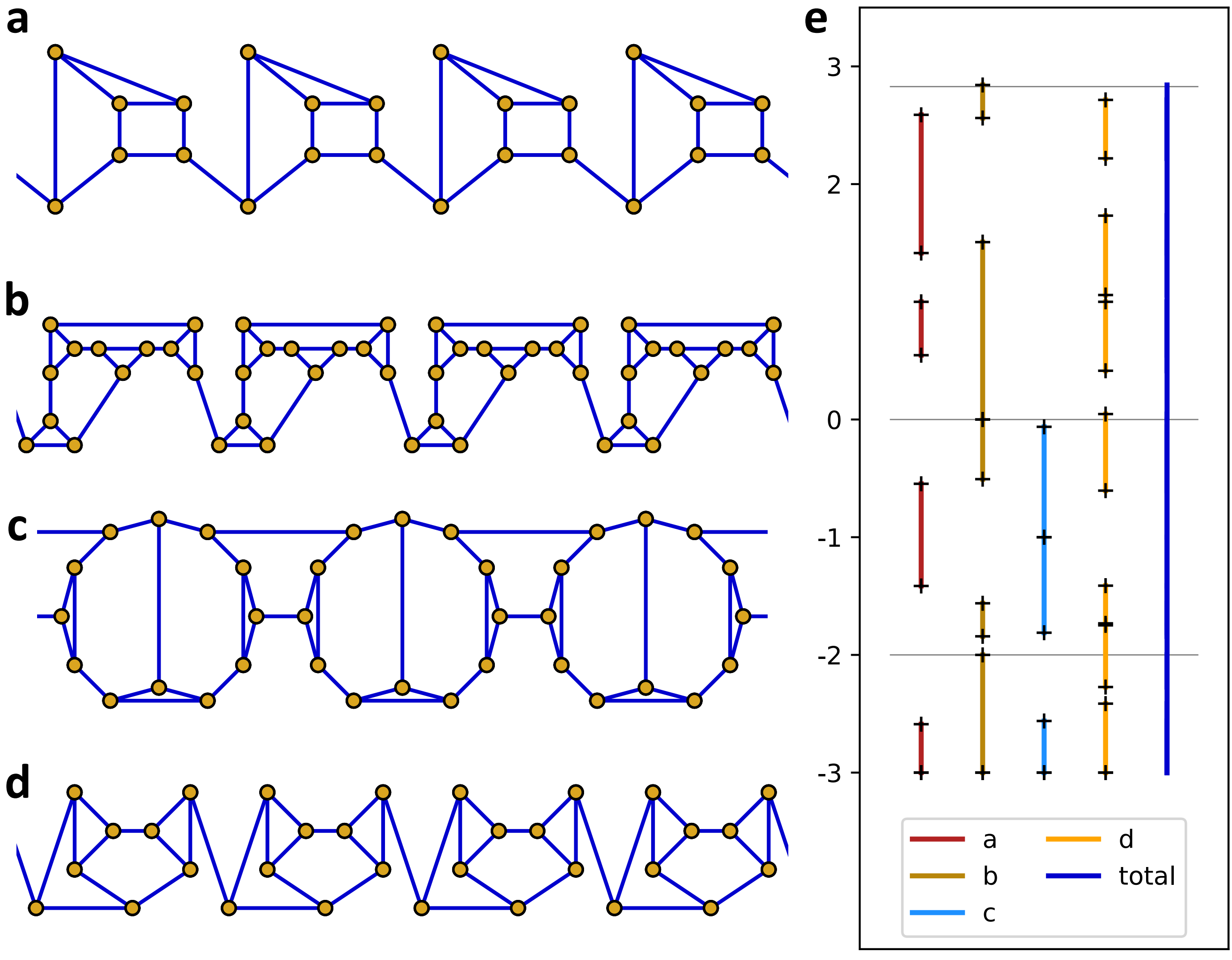
- FINDING SPECIAL CYCLIC (INFINITE) AND $\mathbb{Z}^2$ COVERINGS OF SMALL MEMBERS OF X.

  THESE ARE ANALYZED BY "BLOCH WAVE" OR FLOQUET THEORY.

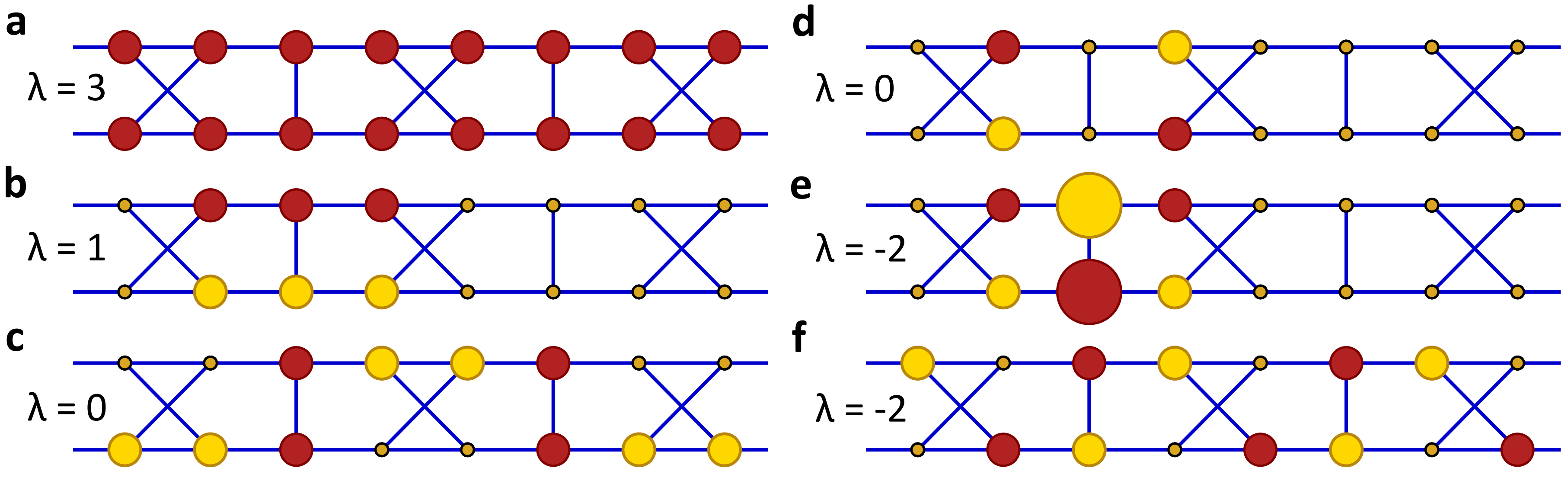
- COMBINATORIAL ~~LY~~ CONSTRUCTION OF APPROXIMATE EIGENFUNCTIONS ON LARGE Y'S TO SHOW MAXIMALITY OF (-2,0) (-1,1).

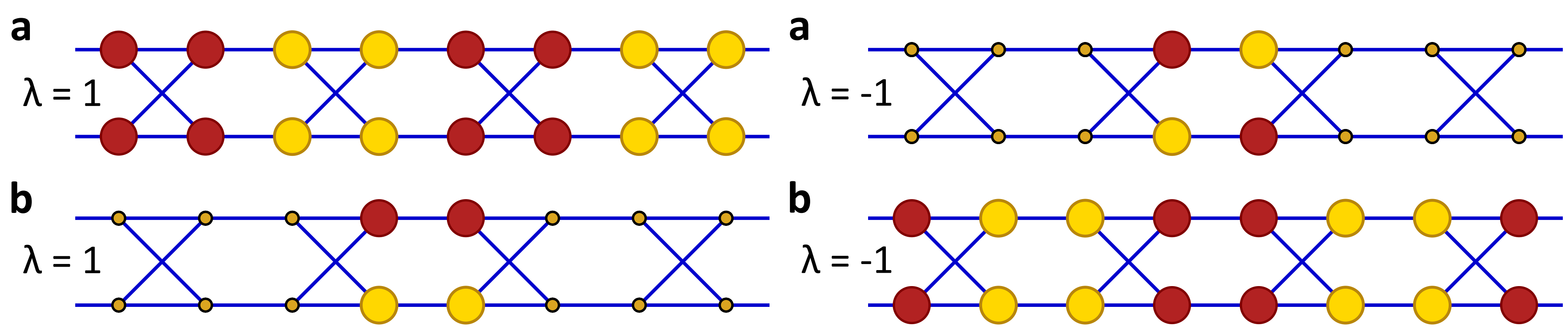


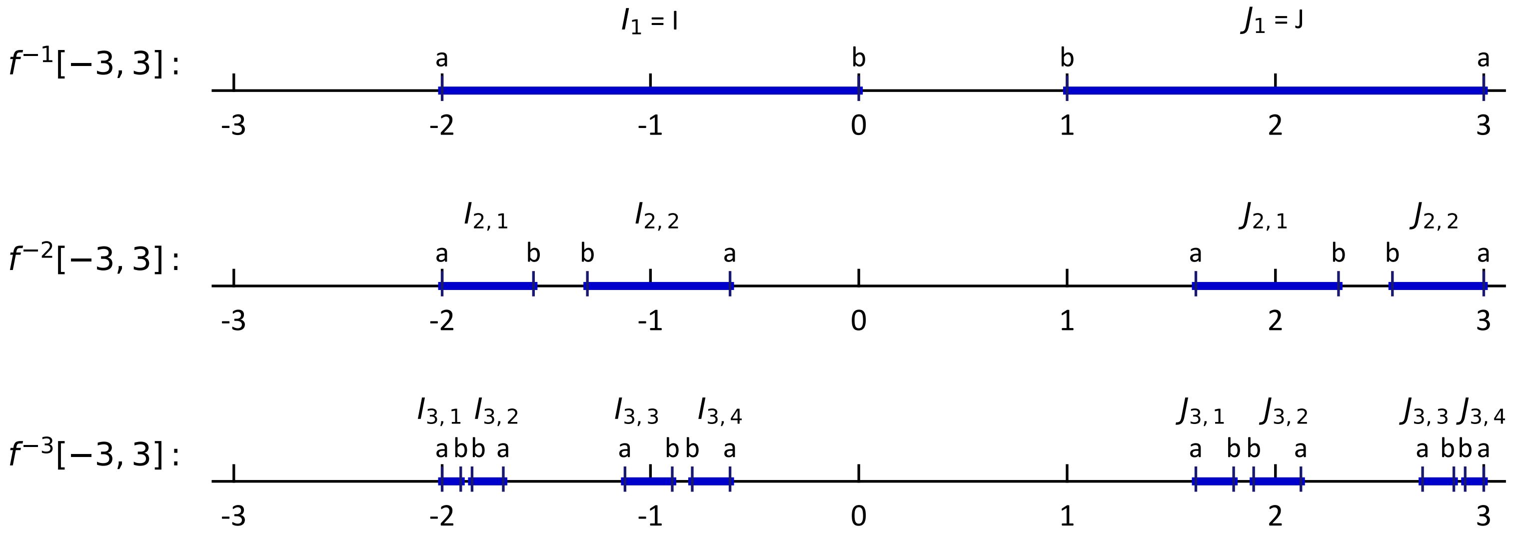
ai
bi
aii
bii
aiii
biii
3
2
0
-2
-3
-1
0
1
θ/2π



a
b
c
d
e
3
2
0
-1
-2
-3
a
b
c
d
total


a
λ = 3
b
λ = 1
c
λ = 0
d
λ = 0
e
λ = -2
f
λ = -2


a
λ = 1
a
λ = -1
b
λ = 1
b
λ = -1



f^{-1}[-3, 3]:
I_1 = I
J_1 = J
a b b a
-3 -2 -1 0 1 2 3
f^{-2}[-3, 3]:
I_{2,1} I_{2,2} J_{2,1} J_{2,2}
a b b a a b b a
-3 -2 -1 0 1 2 3
f^{-3}[-3, 3]:
I_{3,1} I_{3,2} I_{3,3} I_{3,4} J_{3,1} J_{3,2} J_{3,3} J_{3,4}
a b b a a b b a a b b a a b b a
-3 -2 -1 0 1 2 3