Point-wise surjective presentations of stacks, or why I am not afraid of (infinity) stacks anymore.

A. Aizenbud

Weizmann Institute of Science

Joint with Nir Avni

http://www.wisdom.weizmann.ac.il/~aizenr/

Motivation

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This quotient is a groupoid or a (geometric/algebraic) stack.

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- For a group G acting on a space X, define the groupoid $G\backslash X$ to be the groupoid whose set of objects is X and

$$Mor(x, y) = \{g | gx = y\}$$

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s.t. for any scheme S the tuple

$$(Ob(S), Mor(S), s(S), t(S), i(S), inv(S), comp(S))$$

is a groupoid.

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The notion of a functor between two groupoid objects is not local on the source. Namely, if $\mathcal{X} = U_1 \cup U_2$ is an open cover and $\phi_i : U_i \to \mathcal{Y}$ are two functors which become isomorphic when restricted to the intersection, we sometimes cannot find an extension to a functor $\mathcal{X} \to \mathcal{Y}$

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Not all algebraic stacks can be obtained in such a way. In order to obtain all of them, one has to allow Mor to be an algebraic space (and not just a scheme). An algebraic space is an algebraic stack whose automorphism groups are trivial. For this purpose one can use algebraic stacks that are obtained using the construction above.

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Remark

One can replace in the above theorem the field F with any henselian ring with residue field F.

Presentations of a stack by a scheme

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Main steps in the proof

Theorem (A.-Avni 2019)

Let $\mathfrak{X}$ be an algebraic stack. Then for any presentation $X \to \mathfrak{X}$ there is an integer n s.t. for any $x \in \mathfrak{X}(F)$ there is a field extension E/F of degree $\leq n$ s.t. x lies in the essential image of $X(E) \to \mathfrak{X}(E)$,

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- *Moreover, given an integer n one can find a presentation $\phi : X' \to \mathfrak{X}$ s.t. condition (*) will be satisfied for any field extension E/F of degree $\leq n$.*
- *In general, the construction of X' does not give a scheme but only an algebraic space; however, if $\mathfrak{X}$ is an algebraic space and X is affine then X' is a scheme.*

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Corollary

Given 2 diagrams of schemes of the same type, we can define the internal hom between them.

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Let $X' = (X_\bullet \times \mathbb{U}_n)^\wedge_{\mathbb{U}_n} \mathbb{U}_n^\bullet$. The presentation $X' \to \mathfrak{X}$ will satisfy (*) for all field extensions E/F of degree $\leq n$.

Almost surjective presentations

Theorem (A.-Avni 2019)

Let $\mathfrak{X}$ be an algebraic stack. Then, under suitable assumptions on $\mathfrak{X}$ or F, for any surjection $\phi : X \to \mathfrak{X}$ there is an integer n s.t.

$$\mathfrak{X}(F) \subset \bigcup_{\dim_F E \leq n} \phi_E(X(E))$$

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- $\mathfrak{X} = BG$ when $G \to S$ is a flat group scheme over a scheme S.

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$$\mathfrak{X}(F) \subset \bigcup_{\dim_F E \leq n} \phi_E(X(E))$$

Proof.

We prove the theorem by analyzing the following special cases:

- $\mathfrak{X}$ is a scheme
- $\mathfrak{X}$ is an algebraic space
- $\mathfrak{X} = BG$ when $G \to S$ is a flat group scheme over a scheme S.
- $\mathfrak{X} = BG$ when $G \to S$ is a flat group scheme over an algebraic space S.

Almost surjective presentations

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