

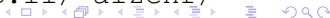
# Point-wise surjective presentations of stacks, or why I am not afraid of (infinity) stacks anymore.

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This quotient is a groupoid or a (geometric/algebraic) stack.



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- For a group  $G$  acting on a space  $X$ , define the groupoid  $G \backslash X$  to be the groupoid whose set of objects is  $X$  and

$$Mor(x, y) = \{g | gx = y\}$$

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s.t. for any scheme  $S$  the tuple

$$(Ob(S), Mor(S), s(S), t(S), i(S), inv(S), comp(S))$$

is a groupoid.

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*The notion of a functor between two groupoid objects is not local on the source. Namely, if  $\mathcal{X} = U_1 \cup U_2$  is an open cover and  $\phi_i : U_i \rightarrow \mathcal{Y}$  are two functors which become isomorphic when restricted to the intersection, we sometimes cannot find an extension to a functor  $\mathcal{X} \rightarrow \mathcal{Y}$*

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## Remark

*One can replace in the above theorem the field  $F$  with any henselian ring with residue field  $F$ .*

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# Main steps in the proof

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*Let  $\mathfrak{X}$  be an algebraic stack. Then for any presentation  $X \rightarrow \mathfrak{X}$  there is an integer  $n$  s.t. for any  $x \in \mathfrak{X}(F)$  there is a field extension  $E/F$  of degree  $\leq n$  s.t.  $x$  lies in the essential image of  $X(E) \rightarrow \mathfrak{X}(E)$ ,*



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- Moreover, given an integer  $n$  one can find a presentation  $\phi : X' \rightarrow \mathfrak{X}$  s.t. condition (\*) will be satisfied for any field extension  $E/F$  of degree  $\leq n$ .
- In general, the construction of  $X'$  does not give a scheme but only an algebraic space; however, if  $\mathfrak{X}$  is an algebraic space and  $X$  is affine then  $X'$  is a scheme.

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- *If  $Y$  is etale over  $S$  then  $X_S^\wedge Y$  is representable by an algebraic space.*
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## Corollary

*Given 2 diagrams of schemes of the same type, we can define the internal hom between them.*



# The main construction

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Let  $X^0 \rightarrow \mathfrak{X}$  be a presentation of a stack over a scheme  $S$ . Let  $S^0 \rightarrow S$  be a finite étale cover.

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Let  $\mathbb{U}_n \subset \mathbb{A}^{n+1}$  be the variety of all monic separable polynomials of degree  $\leq n$ . Let  $\mathbb{U}_n^0 := \{(p, x) \in \mathbb{U}_n \times \mathbb{A}^1 \mid p(x) = 0\}$ .

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# The main construction

## Construction

Let  $X^0 \rightarrow \mathfrak{X}$  be a presentation of a stack over a scheme  $S$ . Let  $S^0 \rightarrow S$  be a finite etale cover.

Let

$$X^n := X^0 \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} X^0$$

and

$$S^n := S^0 \times_S \cdots \times_S S^0$$

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# Almost surjective presentations

## Theorem (A.-Avni 2019)

*Let  $\mathfrak{X}$  be an algebraic stack. Then, under suitable assumptions on  $\mathfrak{X}$  or  $F$ , for any surjection  $\phi : X \rightarrow \mathfrak{X}$  there is an integer  $n$  s.t.*

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- The general case.