Point-wise surjective presentations of stacks, or why I am not afraid of (infinity) stacks anymore.

A. Aizenbud

Weizmann Institute of Science

Joint with Nir Avni

http://www.wisdom.weizmann.ac.il/~aizenr/
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Motivation

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The collection of these objects usually does not depend directly on $G$ and $X$ but only on the "quotient" $G\backslash X$. 
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The collection of these objects usually does not depend directly on $G$ and $X$ but only on the "quotient" $G\backslash X$.

This quotient is a groupoid or a (geometric/algebraic) stack.
Groupoids

**Definition**

A groupoid is a (usually small) category where all morphisms are isomorphisms.

**Examples**

For a group \( G \), define the groupoid \( BG \) to be the groupoid with one object \( pt \) and \( \text{Aut}^{\hat{}}_{pt} \cong G \).

For a group \( G \) acting on a space \( X \), define the groupoid \( G \text{\_\_} f \text{\_\_} X \) to be the groupoid whose set of objects is \( X \) and \( \text{Mor}^{\hat{}}_{x, y} \cong \tilde{g} \mapsto gx \rightarrow y \).
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- For a group $G$, define the groupoid $BG$ to be the groupoid with one object $pt$ and $\text{Aut}(pt) = G$.
- For a group $G$ acting on a space $X$, define the groupoid $G\backslash X$ to be the groupoid whose set of objects is $X$ and

$$\text{Mor}(x, y) = \{g | gx = y\}$$
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- Schemes $\text{Ob}, \text{Mor}$

where the fiber product $\text{Mor} \times \text{Ob}$ is taken w.r.t. the morphisms $\text{t} : \text{Mor} \times \text{Ob}$ and $\text{s} : \text{Mor} \times \text{Ob}$.
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s.t. for any scheme $S$ the tuple

$$(Ob(S), Mor(S), s(S), t(S), i(S), inv(S), comp(S))$$

is a groupoid.
A functor between two groupoid objects \((Ob, Mor)\) and \((Ob', Mor')\) is a pair of morphisms \(Ob \to Ob'\) and \(Mor \to Mor'\) that satisfy some conditions.
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The collection of groupoid objects is a 2-category. By (restricted) Yoneda’s Lemma any groupoid object \(\mathcal{X}\) can be thought of as a functor from the category of schemes to the 2-category of groupoids:

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\mathcal{X}(S) = \text{Funct}(S, \mathcal{X}).
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*The notion of a functor between two groupoid objects is not local on the source.*
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Problem

The notion of a functor between two groupoid objects is not local on the source. Namely, if \(\mathcal{X} = U_1 \cup U_2\) is an open cover and \(\phi_i : U_i \rightarrow \mathcal{Y}\) are two functors which become isomorphic when restricted to the intersection, we sometimes cannot find an extension to a functor \(\mathcal{X} \rightarrow \mathcal{Y}\).
Solution – Stackification.

Definition

Given a groupoid $X$, we can define a functor $X$ from the category of schemes to the 2-category of groupoids in the following way:

The objects of $X$ are given by an (etale) open cover $S = \bigcup_{i} S_i$ and an object of $X$ together with certain compatibility data.

Functors which are obtained in such a way are called algebraic stacks.

Remark

Not all algebraic stacks can be obtained in such a way. In order to obtain all of them, one has to allow $\text{Mor}$ to be an algebraic space (and not just a scheme).

An algebraic space is an algebraic stack whose automorphism groups are trivial. For this purpose one can use algebraic stacks that are obtained using the construction above.
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Let \( \mathcal{X} \) be an algebraic stack. Then \( \mathcal{X} \) can be represented by a groupoid \( \mathcal{X} = (\text{Ob}, \text{Mor}) \) s.t.
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- If $\mathcal{X}$ is QCA, (i.e. the automorphism groups of its object are linear) then the functor $\mathcal{X}(F) \to \mathcal{X}(F)$ is an equivalence for any perfect field $F$. 

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One can replace in the above theorem the field $F$ with any henselian ring with residue field $F$. 

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A morphism $X^0 \to \mathcal{X}$ of a scheme to an algebraic stack is called a presentation, if for any scheme $S$ the morphism $X^0 \times_{\mathcal{X}} S \to S$ is smooth and surjective.
A morphism $X^0 \to \mathfrak{X}$ of a scheme to an algebraic stack is called a presentation, if for any scheme $S$ the morphism $X^0 \times_{\mathfrak{X}} S \to S$ is smooth and surjective.

Given a presentation $X^0 \to \mathfrak{X}$ we can define $X^1 := X^0 \times_{\mathfrak{X}} X_0$. The pair $(X^0, X^1)$ forms a groupoid object.
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Let $X$ be an algebraic stack. Then there is a presentation $X$ s.t. the functor $X^\hat{F}$ is essentially surjective, under suitable assumptions on $X$ or $F$. 

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Let $\mathcal{X}$ be an algebraic stack. Then for any presentation $X \to \mathcal{X}$ there is an integer $n$ s.t. for any $x \in \mathcal{X}(F)$ there is a field extension $E/F$ of degree $\leq n$ s.t. $x$ lies in the essential image of $X(E) \to \mathcal{X}(E)$.
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*Given a presentation $\phi : X \to \mathfrak{X}$ and a field extension $E/F$ we can modify it to a presentation $\psi : X' \to \mathfrak{X}$ s.t.*

$$(*) \psi_F(X'(F)) \supset \phi_E(X(E)) \cap \mathfrak{X}(F)$$
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- Moreover, given an integer $n$ one can find a presentation $\phi : X' \to \mathcal{X}$ s.t. condition (*) will be satisfied for any field extension $E/F$ of degree $\leq n$. 
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- Moreover, given an integer $n$ one can find a presentation $\phi : X' \to \mathcal{X}$ s.t. condition $(*)$ will be satisfied for any field extension $E/F$ of degree $\leq n$.

- In general, the construction of $X'$ does not give a scheme but only an algebraic space; however, if $\mathcal{X}$ is an algebraic space and $X$ is affine then $X'$ is a scheme.
Internal Hom

**Definition**

Let $X$, $Y$ be schemes. Consider the functor $\hat{X}^Y \colon \text{Schemes} \to \text{Sets}$ defined by

$$\hat{X}^Y_S := \text{Hom}^\mathbb{Z}_S(Y, X).$$

More generally, if $X$, $Y$ are $S$-schemes we can define $X^S_Y \colon \text{Schemes} \to S$-sets by

$$X^S_Y_T := \text{Hom}^\mathbb{Z}_S(Y, X).$$

It makes sense to require the following:

- $Y$ is flat over $S$,
- $Y$ is proper over $S$,
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- $Y$ is finite over $S$. 
Proposition

If $Y$ is etale over $S$ then $X$, $S$ $Y$ is representable by an algebraic space. If in addition $X$ is quasi-projective then $X$, $S$ $Y$ is representable by a scheme.

Corollary

Given 2 diagrams of schemes of the same type, we can define the internal hom between them.
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If $Y$ is étale over $S$ then $X^S Y$ is representable by an algebraic space.

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Given 2 diagrams of schemes of the same type, we can define the internal hom between them.
Let $X^0 \to \mathcal{X}$ be a presentation of a stack over a scheme $S$. Let $S^0 \to S$ be a finite etale cover.

Let

$$X^n := X^0 \times \mathcal{X} \cdots \times \mathcal{X} X^0$$

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Let $U_n \subset \mathbb{A}^{n+1}$ be the variety of all monic separable polynomials of degree $\leq n$. 
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Let $U_n \subset \mathbb{A}^{n+1}$ be the variety of all monic separable polynomials of degree $\leq n$. Let $U_n^0 := \{(p, x) \in U_n \times \mathbb{A}^1 | p(x) = 0\}$. 
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Theorem (A.-Avni 2019)

Let $\mathcal{X}$ be an algebraic stack. Then, under suitable assumptions on $\mathcal{X}$ or $F$, for any surjection $\phi : X \to \mathcal{X}$ there is an integer $n$ s.t.

$$\mathcal{X}(F) \subset \bigcup_{\dim_F E \leq n} \phi_E(X(E))$$

Proof. We prove the theorem by analyzing the following special cases:

- $\mathcal{X}$ is a scheme
- $\mathcal{X}$ is an algebraic space
- $\mathcal{X} = BG$ when $G$ is a flat group scheme over a scheme $S$.
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