The relationship between Gini terminology and the ROC curve *

Edna Schechtman†and Gideon Schechtman

E. Schechtman
Department of Industrial
Engineering and Management
Ben Gurion University of the Negev
Beer Sheva, Israel
ednas@bgu.ac.il

G. Schechtman
Weizmann Institute of Science
Department of Mathematics
Rehovot, Israel
gideon@weizmann.ac.il

Abstract

The objectives of this note are to correct a common error and to clarify the connection between the Gini terminology as used in the economic literature and the one used in the diagnostic and classification literature. More specifically, the connection between the area under the receiver operating characteristic (ROC) curve, which is frequently used in the diagnosis and classification literature, and the Gini terminology, which is mainly used in the economic literature, is clarified. It is shown that the area under the ROC curve is related

*Key words: Area under the curve (AUC), Gini covariance, Lorenz curve, Classification
†corresponding author

to the covariance between the two vectors $Y = \{y_i\}_{i=1}^{n_0}$ and $\{i/n_0\}_{i=1}^{n_0}$. Here y_i is the number of items classified to group 1 lying between the $(i-1)^{th}$ and the i^{th} items classified to group 0, and n_0 is the number of items in group 0.

Acknowledgement. We thank Itai Dattner, David Hand, Foster Provost, Benjamin Reiser, Saharon Rosset and Amit Shelef for helpful discussions.

1 Introduction

The receiver operating characteristic (ROC) curve is a graphical plot that illustrates the diagnostic ability of a binary classifier system as its discrimination threshold is varied. The curve is created by plotting the true positive rate (TPR) against the false positive rate (FPR) at various threshold settings. ROC analysis has been used in medicine, radiology, biometrics, forecasting of natural hazards, meteorology, model performance assessment, and other areas for many decades and is increasingly used in machine learning and data mining research.

The relationship between the area under the ROC curve (AUC) and the Gini is noted in several papers. However, the terminology is not always clear, and sometimes it is not consistent with the definitions used in the economic literature. For example, (1999) states: “Note that our usage of the Lorenz curve is actually different from that of economists or demographers. However, we still call it the Lorenz curve... We can also define the Gini index as twice the area between the Lorenz curve itself and the diagonal line”. Lee’s (Lee, 1997; Lee, 1999) interpretation of the Lorenz curve was also used by others. For example, Lilja *et al.*(2011) quotes Lee (1999) and says: “A Lorenz curve plots centiles of the risk factor, in this case PSA, against the cumulative proportion of cases”. Liu, White and Newell (2011) write: “The AUC is also closely related to the Gini coefficient..... which is twice the area between the diagonal and the ROC curve”. Siadaty *et al.*(2004) mention that: “Lee (1999) believes that, using a Lorenz curve, Pietra and Gini indexes have a closer tie with real world medical diagnosis......Correspondence between the suggested method and recent measures of performance like area swept out by the curve, projected length of the curve, or the

Lorenz curve indexes Pietra and Gini may be worth investigating". Recently Irwin and Hautus (2015) treated a special case. They showed that the lognormal model of the Lorenz curve, which is often adopted to model the distribution of income and wealth, is a mirror image of the equal-variance normal model of the ROC curve. They mention that in general ROC curves do not have corresponding rotated Lorenz curves, a fact that was mentioned by Lee (1999) as well.

More commonly, the classification and diagnosis literature that relates the Gini terminology to AUC ignores the difference mentioned in Lee (1999) and wrongly states $Gini = 2AUC - 1$, where the variable for which Gini is computed is not specified. This statement appears in other areas as well. For example, in the Machine Learning literature Hand and Till (2001) write: "The AUC measure of performance is closely related to the Gini coefficient, which is sometimes used as an alternative measure. This is most commonly defined as twice the area between the ROC curve and the diagonal.... Elementary geometry shows that Gini + 1 = 2AUC". The same statement is mentioned by Gajowniczek, Zabkowski and Szupiluk (2014) who write:" Since the ROC curve measures the inequality between the good and the bad score distributions, it seems reasonable to show a relation between the ROC curve and the Lorenz curve. Twice the area between the Lorenz curve and the diagonal line at 45 degree corresponds to the Gini concentration index. This leads to an interesting interpretation of the AUC measure in terms of the Gini coefficient: Gini = 2AUC -1".

We note however that under the additional assumption that the prediction model is well calibrated and unbiased, Wu and Lee (2014) show that the relationship $Gini = 2AUC - 1$ (where Gini is the Gini of the predictive probabilities) holds true.

The objectives of this note are to correct a common error and to clarify the connection between the Gini terminology as used in the economic literature and the one used in the diagnostic and classification literature. The main point is that the standard Gini (sometimes called Gini mean difference (GMD)) is a variability measure of one particular variable, while the AUC involves two variables. It turns out that the economic literature has a term, called *Gini covariance*, which is similar but not quite the same as Gini. In particular, it is a bivariate parameter (while Gini is a univariate parameter that relates X to its cumulative distribution function $F_X(X)$; see Section 2 below), which fits the usage of the diagnostic and classification community perfectly. We show that the ROC curve can be presented as a relative concentration curve for a properly defined X and Y, hence the area between the ROC curve and the 45-degree line can be calculated as a Gini covariance up to constants.

2 AUC in Gini terminology

Gini index is the common measure of inequality: higher value of Gini index indicates higher inequality. We start with the definition of the Gini (usually called Gini mean difference, GMD). The original definition of Gini (by Gini in 1914) is $G(X) = E|X_1 - X_2|$ where X_1 and X_2 are independent and identically distributed copies of X. There are more than a dozen equivalent ways to present Gini (see Yitzhaki, 1998). The way which is most relevant to this note is $G(X) = 4\text{Cov}(X, F_X(X))$, where F_X is the cumulative distribution function of X (see Lerman and Yitzhaki, 1984). A natural extension of the Gini to a bivariate data (which is more relevant to this note) is the Gini-covariance, which is defined for a bivariate variable (X, Y) as $\text{Cov}(X, F_Y(Y))$

where F_Y is the cumulative distribution of Y. Gini covariance is the basic building block in Gini correlation, which is applied when one variable is given in its variate values while the other is ranked (see Schechtman and Yitzhaki, 1987). Note that while Gini is a one dimensional parameter, Gini–covariance (also called co–Gini) is a two-dimensional parameter. Gini index, which is relevant to our paper, is the GMD divided by twice the mean.

Next, we present the Lorenz curve and the concentration curve for the continuous case (see Yitzhaki and Schechtman, 2013). The Lorenz curve plots the cumulative proportion of a variable against the cumulative proportion of the population ranked by it. A typical Lorenz curve will have a convex shape connecting the points $(0, 0)$ with $(1, 1)$. In the example above, the height of the curve above the point $1/4$ is the share of the income of the poorest 25% of the population. The Gini is equal to twice the area bounded between this curve and the diagonal.

Lee (1997) discusses several interpretations of the geometrically defined Gini index. We note in passing that the definition used by him for the Lorenz curve is somewhat different from the one used by economists (Lee, 1997; Lee, 1999). One such interpretation is the coefficient of deviation in disease risk. This coefficient is similar to the coefficient of variation, where the standard deviation is replaced by Gini mean difference.

The concentration curve is the bivariate analogue of the Lorenz curve. It plots the cumulative proportion of one variable against the cumulative proportion of the population ranked by another variable.

Our focus in this paper is on the absolute (ACC) and the relative concentration curves. Therefore we proceed by defining them formally. Let μ_X and μ_Y denote the means of X and Y respectively, and let $f_{Y|X}$ denote the conditional density function of Y given X. The conditional expectation is $g(x) = \mu_{Y.X} = E(Y|X = x)$. It is assumed that all densities are continuous and differentiable, and all second moments exist.

Definition 1 *The absolute concentration curve (ACC) of Y with respect to X, $ACC_{Y.X}(p)$, is implicitly defined by the relationship*

$$ACC_{Y.X}(p) = \int_{-\infty}^{X(p)} g(t) dF_X(t) \tag{1}$$

where $X(p)$ is defined by

$$F_X(X(p)) = p.$$

In words, $X(p)$ is the p^{th} percentile of the distribution of X.

Note that the special case $ACC_{X.X}(p)$ is the absolute Lorenz curve mentioned above. One of the properties of ACC is the following: The area between the diagonal and the ACC is equal to $\mathrm{Cov}(Y, F_X(X))$. That is,

$$\mathrm{Cov}(Y, F_X(X)) = \int_0^1 (\mu_Y p - ACC_{Y.X}(p))dp. \tag{2}$$

The term $\mathrm{Cov}(Y, F_X(X))$ is called *Gini-covariance* of Y and X, or Co-Gini(Y,X). The special case in which $X = Y$ is the Gini of X, up to a constant.

The (relative) concentration curve was probably first defined by Kakwani (1977). It is just the ACC divided by the mean of Y (as will be discussed below, this is more relevant to ROC).

Before we proceed, a few comments are in order:

1. The property above is for the continuous case. As we shall see below the problem dealt with in the classification and diagnosis literature is discrete so some modification of equation (2) is needed. This will be done in Theorems 1 and 2 below.

2. The ACC passes through the points $(0, 0)$ and $(\mu_Y, 1)$, while ROC (and also the relative concentration curve) is plotted in the unit square. (For the properties of ACC, see Yitzhaki and Schechtman, 2013).

3. ROC for a good classifier will lie above the diagonal. Hence the sign in equation (2) should be reversed.

4. The terms to be compared are two estimates of the AUC: $\hat{A}$ (Hand and Till, 2001) as defined in equation (3) below) and $-\text{Cov}(Y, F_X(X)) + 0.5\bar{y}$ (adding the lower triangle), adjusted as per comment 2 above, namely divided by $\bar{y}$.

Following Hand and Till (2001) let $\hat{p}(u)$ be the estimate of the probability that an object with measurement vector u belongs to class 0. Let $f_i = \hat{p}(u_i)$ be the estimated probability of belonging to class 0 for the i^{th} class 0 point from the test set, for $i = 1, ..., n_0$. Define $g_i = \hat{p}(u_i)$ similarly for the n_1 test set points which belong to class 1. Then $(g_1, ..., g_{n_1})$ and $(f_1, ..., f_{n_0})$ are samples from the g and f distributions, respectively. Rank the combined set of values $(g_1, ..., g_{n_1}, f_1, ..., f_{n_0})$ in increasing order. Let r_i be the rank of the i^{th} class 0 test set point. Hand and Till (2001) showed that the AUC can be estimated by

$$\hat{A} = \frac{1}{n_0 n_1} \sum_{i=1}^{n_o} (r_i - i) \tag{3}$$

where n_0 is the number of 0's in the data, n_1 is the number of 1's, r_i is the rank of the i^{th} class 0 test set point in the ranking of the combined data set. There are $(r_i - i)$ class 1 test points with estimated probabilities of belonging to class 0 which are smaller than that of the i^{th} class 0 test point. $\hat{A}$ is the sum of the areas of the rectangles with base length $1/n_0$ and height $(r_i - i)/n_1$.

In their paper, Hand and Till (2001) say that: "this ($\hat{A}$) provides a very straightforward way of estimating the AUC, and one which is immune to errors introduced by smoothing procedures. $\hat{A}$ is equivalent to the test statistic used in the Mann-Whitney-Wilcoxon two sample test, thus demonstrating the equivalence of the AUC and Gini coefficient to this test statistic".

In order to use the Gini terminology, define $t_i = r_i - i$, $i = 1, ..., n_0$, $t_0 = 0$, and note that $0 \leq t_1 \leq t_2 \leq \cdots \leq t_{n_0}$. Let

$$y_i = t_i - t_{i-1} = (r_i - i) - (r_{i-1} - (i-1)).$$

Let Y be a variable taking the values $\{y_i\}_{i=1}^{n_0}$ and T a variable taking the values $\{\tilde{t}_i\}_{i=1}^{n_0}$ (to be defined momentarily) having the joint distribution defined by the following two properties:

(1) If $y_i \neq 0$ then $\text{Prob}((Y,T) = (y_i, \tilde{t}_i)) = \text{Prob}((Y,T) = (y_i, t_i)) = 1/n_0$.

(2) If for some $k < l$ $y_k \neq 0$, $y_{k+1} = \cdots = y_l = 0$ and $y_{l+1} \neq 0$ (equivalently, $t_{k-1} < t_k = \cdots = t_l < t_{l+1}$), then for every $k < i \leq l$ $\text{Prob}((Y,T) = (0, \tilde{t}_i)) = \text{Prob}((Y,T) = (0, t_i + \varepsilon_i)) = 1/n_0$, where $0 < \varepsilon_{k+1} < \varepsilon_{k+2} \cdots < \varepsilon_l < 1$

One should think of the ε_i-s as arbitrarily small numbers. The (admittedly annoying) replacement of some t_i-s by a perturbation of them comes in order to make sure that the distribution of $F_T(T)$ is uniform over the set $\{1/n_0, 2/n_0 \ldots, 1\}$ and in order to keep the formulation of Theorem 1 simple. Otherwise, the formulation should have been changed by replacing $F_Y(Y)$ with another less natural random variable.

As is already implicitly written in (1) and (2) above, $t_i + \varepsilon_i$ is denoted by $\tilde{t}_i$ if i is one of the indices dealt with in case (2) and $\tilde{t}_i = t_i$ if i is in case (1). Note that $\text{Prob}((Y, T) = (y_i, \tilde{t}_i)) = 1/n_0$ and $F_T(\tilde{t}_i) = i/n_0$ for all i. These two facts will be used in the proof of Theorem 1.

Theorem 1

$$\text{Co-Gini}(Y, T) = \text{Cov}(Y, F_T(T)) = \frac{(n_0 + 1)n_1}{2n_0^2} - \frac{1}{n_0^2} \sum_{i=1}^{n_0} t_i. \tag{4}$$

The proof of Theorem 1 is given in the Appendix.

The relationship between the estimate of the AUC as calculated by Hand and Till (2001) and the Gini terminology is stated in the following theorem.

Theorem 2 *With Y and T as defined just before Theorem 1, and $\hat{A}$ as defined in (3),*

$$\hat{A} = -\frac{n_0}{n_1} Cov(Y, F_T(T)) + 0.5 + \frac{1}{2n_0}$$

.

The proof of Theorem 2 is given in the Appendix.

The discrepancy between the two methods of calculation, which is equal to $\frac{1}{2n_0}$, is due to the fact that the data is discrete. The extra factor in the sum-of-rectangles method is the sum of the small triangles that need to be subtracted. Note that when the sample of 0's is big enough, this term is negligible.

Note that since the sequence $y_i = (r_i - i) - (r_{i-1} - (i - 1))$ is neither increasing nor decreasing in general, the ROC curve is neither convex nor concave. In contrast, the Lorenz curve in the economic literature is necessarily convex (there, the y_i is the income of the i-th percentile, hence it is non-decreasing).

3 An example

We illustrate the calculations involved, using an example taken from Vuk and Curk (2006). Table 1 (columns 1 and 2) shows the assigned scores and the real classes of the examples in the given test set. We follow the notation used by [3] and show that the three methods of calculation: Hand and Till (2001), Hanley and McNeil (1982) and the one proposed here give the same result.

INSERT TABLE HERE (now appears on the last page)

The two left-most columns contain the original data: labels and scores (in increasing order). The sample sizes are $n_0 = n_1 = 10$. The area under the ROC curve as evaluated by Hand and Till (2001) is given by the sum of column 7 divided by the product of the sample sizes:

$$\hat{A} = \frac{1}{n_0 n_1} \sum_{i=1}^{n_o} (r_i - i) = 0.68$$

For the Gini-covariance approach, we first calculate the covariance between columns 7 and 8

$$\mathrm{Cov}(y, F_T(t)) = \mathrm{Cov}(y, x) = -0.13.$$

The area under the ROC curve is given by

$$-\frac{n_0}{n_1}\mathrm{Cov}(y, x) + 0.5 + \frac{1}{2n_0} = 0.13 + 0.5 + 0.05 = 0.68.$$

Using Hanley and McNeil (1982), the Mann-Whitney statistic can be calculated as follows:

$$S(X_0, X_1) = 68$$

(the sum of the right-most column), and

$$W = 68/(10 * 10) = 0.68.$$

4 Concluding remarks

We clarify the relation between the terms used by the diagnosis and classification communities and the ones used by the economic community. We show that the AUC is related to the Gini covariance and give the exact formulas for the discrete case. We hope that the connection between ROC and ACC will allow tools developed for ACC to be used by other communities. One such direction may be making inference on dominance of one ROC over another. The interested reader is referred to O'Donnell *et al.* (2008) and the references therein for more detail. Some of the statistical tests involved are available in STATA software.

References

Gajowniczek, K., Zabkowski, T. and Szupiluk, R. (2014). Estimating the ROC curve and its significance for classification models' assessment. Quantitative methods in economics, Vol. XV, 2, 382-391.

Gini, C. (1914). Reprinted: On the measurement of concentration and variability of characters (2005). Metron, LXIII(1), 3-38.

Hand, D.J. and Till, R.J. (2001). A simple generalisation of the area under the ROC curve for multiple class classification problems. Machine Learning, 45, 171–186.

Hanley, J.A. and McNeil, B.J. (1982). The meaning and use of the area under a receiver operating characteristic (ROC) curve. Radiology, 143, 29–36.

Irwin RJ, Hautus MJ. (2015). Lognormal Lorenz and normal receiver operating characteristic curves as mirror images. R. Soc. open sci. 2: 140280.

Kakwani, N.C. (1977). Applications of Lorenz curves in economic analysis. Econometrica, 45, 719–727.

Lee, W.C. (1997). Characterizing exposure-disease association in human populations using the Lorenz curve and Gini index. Stat Med., 16, 729–39.

Lee, W.C. (1999). Probabilistic analysis of global performances of diagnostic tests: interpreting the Lorenz curve-based summary measures. Statist. Med. 18, 455–471.

Lerman, R. I., and Yitzhaki, S. (1984). A note on the calculation and interpretation of the Gini index. Economics Letters, 15, 363–368.

Lilja, H., Cronin, A. M., Dahlin, A., Manjer,J., Nilsson, P.M., Eastham, J.A., Bjartell, A.S., Scardino, P.T., Ulmert, D., and Vickers, A.J.(2011). Prediction of significant prostate cancer diagnosed 20 to 30 years later with a single measure of prostate-specific antigen at or before age 50. Cancer, 117, 1210–1219.

Liu, C., White, M. and Newell, G. (2011). Measuring and comparing the accuracy of species distribution models with presenceabsence data. Ecography, 34, 232–243.

O'Donnell, O., van Doorslaer, E., Wagstaff, A. and Lindelow, M. (2008). Analyzing health equity using household survey data. World Bank Institute, Chapter 7.

Schechtman, E. and Yitzhaki, S. (1987). A measure of association based on Gini's Mean Difference. Communications in Statistics, Theory and Methods, 16(1),207-231.

Siadatya, M.S.,Philbrickb, J.T., Heimc,S.W. and Schectman, J.M. (2004). Repeated-measures modeling improved comparison of diagnostic tests in meta-analysis of dependent studies. Epidemiology, 57, 698–711.

Vuk, M. and Curk, T. (2006). ROC curve, Lift chart and calibration plot. Metodoloski zvezki, 3, 89–108.

Wu, Y.C. and Lee W.C. (2014). Alternative performance measures for prediction models. PLoS ONE, 9(3):e91249.

Yitzhaki, S. (1998). More than a dozen alternative ways of spelling Gini. Research on Economic Inequality, 8, 13–30.

Yitzhaki, S. and Schechtman, E. (2013). The Gini Methodology - a Primer on a Statistical Methodology. Springer.

5 Appendix: proofs of Theorems 1 and 2

Proof of Theorem 1: Recall the definition of $\tilde{t}_i$ given just before the statement of Theorem 1 and denote $x_i = F_T(\tilde{t}_i) = i/n_0$, $i = 1, \ldots, n_0$

$$\mathrm{Cov}(Y, F_T(T)) = \frac{1}{n_0}\sum_{i=1}^{n_0}(x_i - \bar{x})(y_i - \bar{y}) = \frac{1}{n_0}\sum_{i=1}^{n_0} x_i y_i - \frac{n_0+1}{2n_0}\bar{y} \quad (5)$$

We start with the first term of the right-hand side of equation (5).

$$\begin{aligned}\frac{1}{n_0}\sum_{i=1}^{n_0} x_i y_i &= \frac{1}{n_0}\sum_{i=1}^{n_0}(\sum_{j=1}^{i} x_j - \sum_{j=0}^{i-1} x_j) y_i = \frac{1}{n_0}\sum_{i=1}^{n_0}(\sum_{j=1}^{i} x_j - \sum_{j=1}^{i} x_{j-1}) y_i \\ &= \frac{1}{n_0}\sum_{i=1}^{n_0}\sum_{j=1}^{i}(x_j - x_{j-1}) y_i = \frac{1}{n_0}\sum_{j=1}^{n_0}\sum_{i=j}^{n_0}(x_j - x_{j-1}) y_i \\ &= \frac{1}{n_0}\sum_{j=1}^{n_0}(x_j - x_{j-1})\sum_{i=j}^{n_0} y_i = \frac{1}{n_0^2}\sum_{j=1}^{n_0}(\sum_{i=j}^{n_0} y_i) \\ &= \frac{1}{n_0^2}\sum_{j=1}^{n_0}(\sum_{i=1}^{n_0} y_i - \sum_{i=1}^{j-1} y_i) = \bar{y} - \frac{1}{n_0^2}\sum_{j=1}^{n_0}\sum_{i=1}^{j-1} y_i.\end{aligned} \tag{6}$$

So

$$\begin{aligned}cov(Y, F_T(T)) &= \frac{1}{n_0}\sum_{i=1}^{n_0} x_i y_i - \frac{n_0+1}{2n_0}\bar{y} = \bar{y} - \frac{1}{n_0^2}\sum_{j=1}^{n_0}\sum_{i=1}^{j-1} y_i - \frac{n_0+1}{2n_0}\bar{y} \\ &= \frac{n_0-1}{2n_0}\bar{y} - \frac{1}{n_0^2}\sum_{j=1}^{n_0}\sum_{i=1}^{j-1} y_i = \frac{n_0+1}{2n_0^2}\sum_{i=1}^{n_0} y_i - \frac{1}{n_0^2}\sum_{j=1}^{n_0+1}\sum_{i=1}^{j-1} y_i.\end{aligned} \tag{7}$$

We still need to show that

$$\sum_{j=1}^{n_0+1}\sum_{i=1}^{j-1} y_i = \sum_{i=1}^{n_0} t_i.$$

Indeed,

$$\sum_{j=1}^{n_0+1}\sum_{i=1}^{j-1} y_i = \sum_{i=1}^{n_0}(n_0+1-i) y_i = \sum_{i=1}^{n_0}(n_0+1-i)(t_i - t_{i-1})$$

$$= \sum_{i=0}^{n_0}(n_0+1-i) t_i - \sum_{i=0}^{n_0-1}(n_0-i) t_i = \sum_{i=0}^{n_0-1} t_i + t_{n_0} = \sum_{i=1}^{n_0} t_i. \qquad \blacksquare$$

Proof of Theorem 2:

Note that $\sum_{i=1}^{n_0} y_i = n_1$, so $\bar{y} = n_1/n_0$ and by Theorem 1,

$$-\mathrm{Cov}(Y, F_T(T)) + 0.5\bar{y} = \frac{1}{n_0^2}\sum_{i=1}^{n_0} t_i - \frac{(n_0+1)n_1}{2n_0^2} + 0.5\bar{y} = \frac{1}{n_0^2}\sum_{i=1}^{n_0} t_i - \frac{n_1}{2n_0^2}.$$

Dividing both sides by $\bar{y} = \frac{n_1}{n_0}$ we get

$$-\frac{n_0}{n_1}\mathrm{Cov}(Y, F_T(T)) + 0.5 = \frac{\sum_{i=1}^{n_0} t_i}{n_1 n_0} - \frac{1}{2n_0} = \hat{A} - \frac{1}{2n_0}.$$

where the last equation follows (3). This completes the proof. ■

1	2	3	4	5	6	7	8	9	10	11
Labels	Scores	f_i & g_i by labels	f_i	g_i	r_i	$t_i = r_i - i$	$\tilde{t}_i$	$X_i = F(\tilde{t}_i)$	$Y_i = t_i - t_{i-1}$	# 1-scores less than 0-score
1	0.1	g_1	0.3	0.1	3	2	2	0.1	2	-
1	0.2	g_2	0.47	0.2	7	5	5	0.2	3	-
0	0.3	f_1	0.48	0.4	8	5	5.1	0.3	0	2
1	0.4	g_3	0.495	0.45	10	6	6	0.4	1	-
1	0.45	g_4	0.61	0.46	12	7	7	0.5	1	-
1	0.46	g_5	0.63	0.49	14	8	8	0.6	1	-
0	0.47	f_2	0.64	0.6	15	8	8.1	0.7	0	5
0	0.48	f_3	0.65	0.62	16	8	8.2	0.8	0	5
1	0.49	g_6	0.67	0.66	18	9	9	0.9	1	-
0	0.495	f_4	0.9	0.7	20	10	10	1	1	6
1	0.6	g_7	-	-	-	-	-	-	-	-
0	0.61	f_5	-	-	-	-	-	-	-	7
1	0.62	g_8	-	-	-	-	-	-	-	-
0	0.63	f_6	-	-	-	-	-	-	-	8
0	0.64	f_7	-	-	-	-	-	-	-	8
0	0.65	f_8	-	-	-	-	-	-	-	8
1	0.66	g_9	-	-	-	-	-	-	-	-
0	0.67	f_9	-	-	-	-	-	-	-	9
1	0.7	g_{10}	-	-	-	-	-	-	-	-
0	0.9	f_{10}	-	-	-	-	-	-	-	10

Table 1: An illustrative example