The number of closed ideals in the algebra of bounded operators on Lebesgue spaces

Gideon Schechtman

Abstract

We survey some of the recent progress in determining the number of two sided closed ideals in the Banach algebras of bounded linear operators on Lebesgue spaces, $L_p[0,1]$. In particular we discuss two recent results: The first of Johnson, Pisier and the author showing that there are a continuum of such ideal in the case of $p = 1$. The second a result of Johnson and the author showing that in the case $1 < p < \infty$, $p \neq 2$, there are exactly 2 to the continuum such ideals.

Published by EMS Press
DOI 10.4171/ICM2022/?

1. Introduction

For a Banach space X, over the real or complex field, we denote by $L(X)$ the Banach algebra of bounded linear operators on X. The wider subject of study here is the structure of the class of closed two sided ideals in this algebra. We recall that a closed ideal here is a closed linear subspace M of $L(X)$ such that if $T \in M$ and $A, B \in L(X)$ then $ATB \in M$. The research we shall concentrate on describing here is concerned with the modest aim of deciding what is the number of such different closed ideals when X is one of the Lebesgue spaces, $L_p[0,1]$, $1 \le p \le \infty$.

Recall that for a measure space $(\Omega, \mathcal{F}, \mu)$ and $1 \le p < \infty$, $L_p(\Omega, \mathcal{F}, \mu)$ denotes the Banach spaces of all (equivalence classes of) $\mathcal{F}$ measurable functions, f, such that $\|f\|_p = (\int_\Omega |f|^p d\mu)^{1/p} < \infty$. $L_\infty(\Omega, \mathcal{F}, \mu)$ denotes the space of essentially bounded functions with the sup norm. Of particular interest will be the case of $\Omega = \mathbb{N}$ with the counting measure in which case we'll denote the space by ℓ_p and the case when $(\Omega, \mathcal{F}, \mu)$ is the interval $[0,1]$ with Lebesgue measure which we'll denote by $L_p[0,1]$. We recall that for $1 \le p < \infty$, any infinite dimensional separable $L_p(\Omega, \mathcal{F}, \mu)$ space is isomorphic to either $L_p[0,1]$ or ℓ_p. Of course $L_2[0,1]$ and ℓ_2 are isometric. Also, $L_\infty[0,1]$ and ℓ_∞ are known to be isomorphic. We also denote by c_0 the subspace of ℓ_∞ of sequences tending to zero.

Most probably the first result in concerning the structure of ideals of $L(X)$ is the influential work of Calkin [7] who showed that if X is a separable Hilbert space then the only non trivial (that is, different from the whole space and $\{0\}$) closed ideal here is the ideal of compact operators. This result was generalized in [11] to the other separable classical sequence spaces ℓ_p, $1 \le p < \infty$, and c_0. There were more results for special cases including some natural non separable ones. Pietsch's book [17, Chapter 5] contains a survey of the results obtained by 1980. Pietsch also points out that, following a known result, in $L(L_p[0,1])$, $1 < p < \infty$, $p \ne 2$, there are countably many different closed ideals and raises the question of how many ideals are there in $L(L_p[0,1])$, $1 \le p < \infty$, and some other classical spaces.

The subject of the structure of the set of ideals in $L(X)$ lay dormant for a while but gain new drive since the beginning of this century. We shall not survey most of these developments. We refer to the very good introduction in [3] for a survey of the known results up to a couple of years ago. Let us just say that there are very few spaces X for which we have a complete knowledge of all of the ideals in $L(X)$. From now on we shall concentrate only on the question of the number of closed ideals in $L(X)$ for X being a separable L_p or some related space.

Note that if P is a (always bounded, linear) projection onto a subspace Y of X and Y is isomorphic to its square, $Y \oplus Y$,then the set

$$\{APB;\ A, B \in L(X)\}$$

is a closed ideal in $L(X)$. This is easy to verify. (The requirement that Y is isomorphic to its square comes to ensure that this set is a closed subspace of $L(X)$.) If P_1, P_2 are two such projections onto Y_1, Y_2 respectively and if there is no isomorphism of X which carries Y_1 onto Y_2 then the two ideals generated are different. In particular this is the case if Y_1 and Y_2 are not isomorphic. By the time [17] was written, it was known, [19], that there are countably many mutually non isomorphic complemented subspaces (i.e., ranges of projections) of $L_p[0,1]$, $1 < p < \infty$, $p \neq 2$, each isomorphic to its square (in particular they are infinite dimensional). So the reasoning above, appearing in [17], yields infinitely many different closed ideals in $L_p[0,1]$, $1 < p < \infty$, $p \neq 2$. Pietsch asked in his book whether there are uncountably many such ideals and also what is the number of closed ideals in $L_1[0,1]$ (in which at the time only finitely many closed ideals were known). Shortly afterward, [6] produced $\aleph_1$ mutually non-isomorphic complemented subspaces of $L_p[0,1]$ each isomorphic to its square, raising the number of different ideals in $L(L_p[0,1])$ to $\aleph_1$. In his book Pietsch also noticed (building again on known complemented subspaces) that in the algebra of bounded operators on $C(0,1)$, the space of continuous functions on the unit interval, there are $\aleph_1$ different closed ideals. We remark in passing that in the situation above (where the different ideals in each case are related to complemented subspaces) the ideals constructed are not even mutually isomorphic as Banach algebras. This follows immediately from a result of Eidelheit [8]: If Y and Z are Banach spaces such that the algebras $L(Y)$ and $L(Z)$ are isomorphic as Banach algebras then Y and Z are isomorphic as Banach spaces (and trivially vice versa).

The main purpose of this note is to focus on two recent advancements in this direction in which the author was involved. In [13] we built a continuum of different closed ideals in $L(L_1[0,1])$, in $L(C(0,1))$ and also in $L(L_\infty[0,1])$. (In these results and also in the other ones we survey here we are not using the continuum hypothesis so the cardinality of the continuum may be larger than $\aleph_1$.) The result is stated as Theorem 5.1 below.

The result of [15] may be more surprising: The number of different closed ideals in each of $L(L_p[0,1])$, $1 < p < \infty$, $p \neq 2$, is $2^{\mathfrak{c}}$ ($\mathfrak{c} = 2^{\aleph_0}$ is the cardinality of the continuum). The upper bound is simple. The problem is to produce $2^{\aleph_0}$ ideals.The result is stated as Theorem 4.1 below.

The proofs of the two results are quite different but a common feature is that the constructions and proofs (that the constructed ideals are really different) boils down to inequalities in the quantitative finite dimensional world and involve probabilistic and/or harmonic analytical methods.

Except for these two papers there are more results, by others, involving related questions of Pietsch, that we shall only report on here but will not go into the detailed constructions. Pietsch asked whether for $1 \le p < q < \infty$, $L(\ell_p \oplus \ell_q)$ contains infinitely many different closed ideals. This was solved by Schlumprecht and Zsák [20] producing a continuum of such ideals in these spaces as well as in $L(\ell_p \oplus c_0)$, $1 \le p < \infty$. Later, building in part on the method in [15], Freeman, Schlumprecht and Zsák in [9], [10] showed that there are $2^{\mathfrak{c}}$ different closed ideals in the spaces $L(\ell_p \oplus \ell_q)$, $1 < p < q < \infty$ as well as in $L(\ell_p \oplus c_0)$, $L(\ell_p \oplus \ell_\infty)$, $L(\ell_p \oplus \ell_1)$, $1 < p < \infty$.

In all the questions and answers described above, two ideals are considered different if they are different as sets. There are of course weaker distinctions one can consider. A natural one is to consider two ideals to be different if they are not isomorphic as Banach algebras; i.e., are not homomorphic by an homomorphism which is continuous in both directions. In Corollary 6.7 we report on a recent observation of Bill Johnson, Chris Phillips and the author, based of Eidelheit's [8], showing that this seemingly weaker distinction still gives the same results.

Another question from [17] was whether ℓ_p, $1 \le p < \infty$ and c_0 are the only spaces X in which the only non-trivial closed ideal in $L(X)$ is the ideal of compact operators. This turned out not to be the case: Solving an old problem of Lindenstrauss, Argyros and Haydon [1] built a Banach space in which every operator is a multiple of the identity plus a compact operator from which it easily follows that the only non trivial closed ideal in the space of operators on this space is the ideal of compact operators.

In Section 2 we survey what was previously known about closed ideals in the space of operators on separable L_p spaces, $1 \le p < \infty$. We also define the notions of small and large ideals. Section 3 deals with a criterion for Banach spaces X ensuring the existence of $2^{\mathfrak{c}}$ different closed ideals which turns out to be relevant for a construction of that many ideals in $L(L_p[0,1])$, $1 < p < \infty$. The criterion is in terms of the existence of a certain operator on the space X. Section 4 is devoted to the construction of such an operator. Here the presentation is different than in the original paper and, since we think it may be useful in the future, is given in more detail. Section 5 deals with the case of $L(L_1[0,1])$ and is independent of the previous sections. In the final section we gather some remarks and open problems.

2. Old ideals

Here we survey what was known previous to [20], [13] and [15]. It is not needed for reading the next 3 sections which contain newer results. We begin with a few simple observations about (always two sided) closed ideals in $L(X)$ for a general (infinite dimensional) Banach space X. Since any Banach space admits a rank one operator and since for any two rank one operators $R_1, R_2 \in L(X)$ there are $S, T \in L(X)$ with $R_2 = SR_1T$, every ideal in $L(X)$ contains any rank one operator and, since it is a subspace, all finite rank operators. So any closed ideal in $L(X)$ contains the closure of the finite rank operators, $\mathcal{F}(X)$. If X has the approximation property, as any L_p space and all the other classical Banach spaces have, then $\mathcal{F}(X)$ is equal to the ideal of compact operators, $\mathcal{K}(X)$. Since X is infinite dimensional $\mathcal{K}(X)$ is a proper ideal. As we already mentioned, for some X including ℓ_p, $1 \le p < \infty$, and c_0, $\mathcal{K}(X)$ is the only proper closed ideal in $L(X)$. Another closed "small" proper ideal presented in every space (although sometimes coincides with the compact operators) is that of the strictly singular operators, $\mathcal{S}(X)$; i.e., the set of all operators on $L(X)$ which are not an isomorphism when restricted to any infinite dimensional subspace. We call a closed ideal in $L(X)$ *small* if it's contained in $\mathcal{S}(X)$ otherwise we call it *large*. This distinction is not always useful but in $L(L_p[0,1])$ and in particular in $L(L_1[0,1])$ it contributes to the understanding of the structure of the class of closed ideals as we shall see. It also gives rise to some open problems.

In $L(L_1[0,1])$ consider the set $\mathcal{I}_{\ell_1}$ of operators that factor through ℓ_1. It turns out that this is a closed ideal. It is of course large (as ℓ_1 is isometric to a complemented subspace of $L_1[0,1]$). It follows from known results (see the introduction in [13] for this and other unexplained reasoning in this section concerning $L(L_1[0,1])$) that I_{ℓ_1} contains $\mathcal{S}(L_1[0,1])$ and is contained in any large ideal. In particular any large ideal in $L(L_1[0,1])$ contains any small ideal.

Except for $\mathcal{K}(L_1[0,1])$ and $\mathcal{S}(L_1[0,1])$ there is another classical small ideal: This is the set of Danford–Pettis operators (operators which send weakly compact sets onto norm compact sets). As for classical large ideals, except for $\mathcal{I}_{\ell_1}$ there is only one other known large ideal in $L(L_1[0,1])$. This is the maximal proper ideal which turns out to be the same as all operators which are not isomorphism when restricted to a subspace of $L_1[0,1]$ isomorphic to $L_1[0,1]$ (the fact that this is an ideal is not trivial at all). The continuum of ideals produced in Section 5 are all small. We do not know if there are infinitely many large ideals in $L(L_1[0,1])$.

For $L(L_p[0,1])$, $1<p<\infty$, $p\neq 2$, the break point between small and large ideals is not as sharp as for $p=1$. Except for the ideal $\mathcal{I}_{\ell_p}$ of all operators which factor through ℓ_p there is another incomparable minimal large ideal. This is the closure of the operators which factor through ℓ_2, denoted $\overline{\Gamma_2}$. It turns out that every large ideal contains one of these two ideals. However, $\overline{\Gamma_2}$ does not contain the strictly singular operators in $L(L_p[0,1])$, $1<p<\infty$, $p\neq 2$. So not every large ideal contains all the small ideals. We refer to the introduction in [15] for this and other unexplained reasoning here. As was already remarked above there were $\aleph_1$ large closed ideals known in $L(L_p[0,1])$ for quite a while (an ideal generated by a projection onto an infinite dimensional complemented subspace is clearly large). There is also the maximal ideal of all operators not preserving an isomorphic copy of $L_p[0,1]$ which is clearly large. (Again the fact that this is an ideal is not simple). [20] produced a continuum of small ideals in these $L(L_p[0,1])$ algebras. Previous to [20] there were only a finite number of such small ideals. As is exposed in Sections 3 and 4 below, in [15] we produced $2^{\mathfrak{c}}$ large ideals as well as $2^{\mathfrak{c}}$ small ideals in $L(L_p[0,1])$, $1<p<\infty$, $p\neq 2$.

3. A criterion for having many closed ideals

This section is taken almost verbatim from [15, Section 2].

Recall first the notion of unconditional basis for a Banach space X. A sequence $\{e_i\}_{i=1}^\infty$ is said to be a (Schauder) basis for X if any $x\in X$ has a unique representation as $x=\sum_{i=1}^\infty a_ie_i$ for some coefficients $\{a_i\}$, The basis $\{e_i\}_{i=1}^\infty$ is said to be K-*unconditional* if for all signs $\{\varepsilon_i\}_{i=1}^\infty\in\{-1,1\}^{\mathbb{N}}$ and all $\sum_{i=1}^\infty a_ie_i\in X$,

$$\|\sum_{i=1}^\infty \varepsilon_i a_i e_i\|\le K\|\sum_{i=1}^\infty a_ie_i\|.$$

Note that given a subset $\mathbb{M}$ of $\mathbb{N}$, the natural projection $P_{\mathbb{M}}$, given by $P_{\mathbb{M}}(\sum_{i=1}^\infty a_ie_i)=\sum_{i\in\mathbb{M}}a_ie_i$, is of norm at most K. We also denote its range, the closed linear span of $\{e_i\}_{i\in\mathbb{M}}$ by $[e_i]_{i\in\mathbb{M}}$. It is also true and easy to show that an unconditional basis is a Schauder basis in any order.

The theorem below is stated for a 1-unconditional basis, enough for our purposes, but can easily be generalized for any K-unconditional basis.

There is a continuum of infinite subsets of the natural numbers $\mathbb{N}$, each two of which have only a finite intersection. Denote some fixed such continuum by C. For a finite dimensional normed space E, we denote by $d(E)$ the Banach–Mazur distance of E to an Euclidean space, i.e., if the dimension of E is k then

$$d(E) = \inf\{\|A\|\|B\| \; ; \; A : \ell_2^k \to E,\; B : E \to \ell_2^k, AB = I_E\}.$$

Also, recall that for an operator $T : X \to Y$ between two normed spaces, $\gamma_2(T)$ denotes its factorization constant through a Hilbert space:

$$\gamma_2(T) = \inf\{\|A\|\|B\| \; ; \; A : H \to Y, B : X \to H,\; T = AB,\; H \text{ a Hilbert space}\}.$$

If T is of rank k, then $\gamma_2(T) \leq k^{1/2}\|T\|$ because every k dimensional normed space is $k^{1/2}$-isomorphic to ℓ_2^k. Note that $d(E)$ is just $\gamma_2(I_E)$, where I_E is the identity operator on E.

Theorem 3.1. *Let X be a Banach space with a 1-unconditional basis $\{e_i\}$, let Y be a Banach space, and let $T : X \to Y$ be an operator of norm at most one satisfying:*

(a) For some $\eta > 0$ and for every M there is a finite dimensional subspace E of X such that $d(E) > M$ and $\|Tx\| > \eta\|x\|$ for all $x \in E$.

(b) For some constant Γ and every m there is an n such that every m-dimensional subspace E of $[e_i]_{i \geq n}$ satisfies $\gamma_2(T_{|E}) \leq \Gamma$.

Then there exist natural numbers $1 = p_1 < q_1 < p_2 < q_2 < \ldots$ such that, denoting for each k, $G_k := [e_i]_{i=p_k}^{q_k}$, and defining for each $\alpha \in C$, the operator $P_\alpha : X \to [G_k]_{k\in\alpha}$ to be the natural basis projection, and setting $T_\alpha := TP_\alpha$, we have the following:

If $\alpha_1, \ldots, \alpha_s \in C$ (possibly with repetitions) and $\alpha \in C \setminus \{\alpha_1, \ldots, \alpha_s\}$, then for all $A_1, \ldots, A_s \in L(Y)$ and all $B_1, \ldots, B_s \in L(X)$,

$$\|T_\alpha - \sum_{i=1}^{s} A_i T_{\alpha_i} B_i\| \geq \eta/2. \tag{3.1}$$

Since we don't have anything to add to the original proof of this theorem we refer the interested reader to [15] for the not-so-hard proof.

Theorem 3.1 provides a criterion for having $2^{\mathfrak{c}}$ different closed ideals in a space satisfying the assumptions of the theorem.

Corollary 3.2. *Let X be a Banach space with a 1-unconditional basis $\{e_i\}$ and assume there is an operator $T : X \to X$ of norm at most one satisfying (a) and (b) of Theorem 3.1. Then $L(X)$ has exactly $2^{\mathfrak{c}}$ different closed ideals.*

Proof. Indeed, for any nonempty proper subset $\mathcal{A}$ of C let $\mathcal{I}_\mathcal{A}$ be the ideal generated by $\{T_\alpha\}_{\alpha\in\mathcal{A}}$; i.e., all operators of the form $\sum_{i=1}^{s} A_i T_{\alpha_i} B_i$ with $s \in \mathbb{N}$, $A_i, B_i \in L(X)$, $\alpha_i \in \mathcal{A}$, $i = 1, \ldots, s$. Since we allow repetition of the T_{α_i}, it is easy to see that this really defines a (non closed) ideal. We'll show that, when $\mathcal{A}$ ranges over the non empty proper subsets of C, $\overline{\mathcal{I}_\mathcal{A}}$ define different closed ideals.

Let $\mathcal{B}$ be a subset of C different from $\mathcal{A}$ and assume, without loss of generality, that $\mathcal{B} \not\subset \mathcal{A}$. Let $\alpha \in \mathcal{B} \setminus \mathcal{A}$. Then by Theorem 3.1, $T_\alpha \notin \overline{\mathcal{I}_{\mathcal{A}}}$. Consequently, $\overline{\mathcal{I}_{\mathcal{A}}}$ and $\overline{\mathcal{I}_{\mathcal{B}}}$ are different.

Since the density character of $L(X)$, for any separable X, is at most the continuum, it is easy to see that, for any separable space X, $L(X)$ has at most $2^{\mathfrak{c}}$ different closed ideals. ∎

Remark 3.3. If Y is a Banach space that contains a complemented subspace X with the properties of Corollary 3.2 then clearly $L(Y)$ also has $2^{\mathfrak{c}}$ different closed ideals. The same is true also for any space isomorphic to such a Y. Also, the assumption that T has norm one can be weakened to just requiring that T is bounded.

Remark 3.4. By the discussion just before Corollary 6.7 below, if Y is as in the previous remark then $L(Y)$ actually has $2^{\mathfrak{c}}$ closed ideals each two of which are not isomorphic as Banach algebras. That is, there is no homomorphism between them which is continuous in both directions.

Maybe the simplest examples of spaces X that satisfy the hypotheses of Corollary 3.2 (and thus $L(X)$ has $2^{\mathfrak{c}}$ different closed ideals) are $(\sum \ell_{r_i}^{n_i})_2$ for $r_i \uparrow 2$ and n_i satisfying $n_i^{\frac{1}{r_i}-\frac{1}{2}} \to \infty$. These spaces satisfy the assumptions with T being the identity. To verify (a) is simple with E being one of the spaces $\ell_{r_i}^{n_i}$ for i large enough. To verify (b) is a bit more involved and, since as we shall shortly remark that this space is not good for our purposes, we shall not enter into the reasoning here. (The main point is that the distance of the worst m dimensional subspace of L_r from an Euclidean space tends to 1 when r tends to 2.) Unfortunately $(\sum \ell_{r_i}^{n_i})_2$ for $r_i \uparrow 2$ and $n_i^{\frac{1}{r_i}-\frac{1}{2}} \to \infty$ does not embed isomorphically as a complemented subspace into any L_p, $p < \infty$, so this example is not good for our purposes. Actually, at least for some sequences $\{(r_i, n_i)\}$ with the above properties, $(\sum \ell_{r_i}^{n_i})_2$ does not even embed isomorphically into any L_p space, $p < \infty$. That this is true, for example, if each $(r, n) \in \{(r_i, n_i)\}$ repeats n times follows from Corollary 3.4 in [16].

In the next section we show how to get complemented subspaces of the reflexive L_p spaces that satisfy the hypotheses of Corollary 3.2.

4. A special operator and the case of reflexive Lebesgue spaces

In order to apply the criterion in Theorem 3.1 and deduce by Corollary 3.2 the existence of $2^{\mathfrak{c}}$ different closed ideals in $L_p[0,1]$, $1 < p < \infty$, $p \neq 2$, it is enough, by Remark 3.3, to find a complemented subspace of a space isomorphic to $L_p[0,1]$ having a 1-unconditional basis and an operator on it satisfying (a) and (b) of Theorem 3.1. In [15] this is done by using a certain complemented subspace of $L_p[0,1]$, $1 < p < \infty$, $p \neq 2$, and a certain operator on it (which is a variant of an operator the authors used in a previous paper [14] for a different purpose). The complemented subspace, X_p, is a span of independent, 3-values, symmetric random variables. The space X_p which was investigated by Rosenthal starting with [18] was very influential in studying the geometry of L_p spaces. The operator is a certain diagonal operator between two such X_p spaces (followed by an injection of the second space into the first). This is where probabilistic inequalities, alluded to in the introduction, enter into the reasoning.

Here we shall describe the construction in a different way (although if one digs into the roots of the two constructions they amount to basically the same operator). We think the presentation here may be cleaner and thus more accessible for further applications.

We begin with a non traditional representation of (a space isomorphic to) $L_p[0,1]$. For $2 < p \leq \infty$ define M_p to be $L_p(0,\infty) \cap L_2(0,\infty)$ with norm

$$\|f\|_{M_p} = \max\{\|f\|_{L_p(0,\infty)}, \|f\|_{L_2(0,\infty)}\}.$$

For $1 \leq q < 2$ we define M_q to be $L_q(0,\infty) + L_2(0,\infty)$ with norm

$$\|f\|_{M_q} = \inf\{\|g\|_{L_q(0,\infty)} + \|h\|_{L_2(0,\infty)} \ ; \ f = g + h\}.$$

Here, $L_r(0,\infty)$ denotes the space of functions, f, on $(0,\infty)$ with $\|f\|_r = (\int_0^\infty |f(t)|^r\, dt)^{1/r} < \infty$. ($M_2 := L_2(0,\infty)$)

Note that M_p, $1 \leq p \leq \infty$ are rearrangement invariant spaces; i.e., the norm of f depends only on the distribution of $|f|$. Also it is easy to prove that for $1 \leq p < \infty$, the dual of M_q is M_p where, $\frac{1}{q} + \frac{1}{p} = 1$. In [12, Chapter 1] it is proved that, for $1 < p < \infty$, $p \neq 2$, M_p is isomorphic to $L_p[0,1]$. We remark in passing that this is done based on Rosenthat's [18] and is where probabilistic inequalities are used; In the presentation below probability will not show anymore. So, $L_p[0,1]$, $1 < p < \infty$, $p \neq 2$, has two different isomorphic representations as rearrangement invariant function spaces on $(0,\infty)$. It is also proved in [12] that these are the only two such representations, a fact we'll not use here. For $p = 1$ and $p = \infty$, M_p is not isomorphic to $L_p[0,1]$.

If $q < r < 2$ then the function $f_r(t) = t^{-1/r}$ is in M_q. Indeed,

$$\|f_r\|_{M_q} \leq \|f_r \mathbf{1}_{(0,1)}\|_q + \|f_r \mathbf{1}_{1,\infty)}\|_2 < \infty.$$

If $f^1, f^2, \ldots$ are disjoint functions on $(0,\infty)$, each (when restricted to its support) with the same distribution as f_r, then $\{f^i\}_{i=1}^\infty$ is isometrically equivalent in M_q to the unit vector basis of ℓ_r. This actually holds in any rearrangement invariant function space on $(0,\infty)$ containing the function f_r and follows from the simple fact that if $\sum_{i=1}^\infty |a_i|^r = 1$ then $|\sum_{i=1}^\infty a_i f^i|$ has the same distribution as f_r.

For $1 \leq q < r < 2$ and $s > 1$ define $D_s : M_q \to M_q$ by $D_s f(t) = s^{1/r} f(st)$. Note that

$$D_s f_r = f_r, \tag{4.1}$$

for all $f \in L_2(0,\infty)$

$$\|D_s f\|_2 = s^{\frac{1}{r} - \frac{1}{2}} \|f\|_2, \tag{4.2}$$

and for all $f \in L_q(0,\infty)$

$$\|D_s f\|_q = s^{\frac{1}{r} - \frac{1}{q}} \|f\|_q. \tag{4.3}$$

Also, $D^*: M_p \to M_p$, $p = q/(q-1)$, is given by

$$D_s^* g(t) = s^{\frac{1}{r}-1} g(t/s). \tag{4.4}$$

Given $0 < \delta < 1$, put $s = s(\delta) = \delta^{\frac{rq}{q-r}}$ and define $r = r(\delta)$ by $s^{\frac{1}{r}-\frac{1}{2}} = 2$. Note that $\delta \searrow 0$ implies that $s(\delta) \nearrow \infty$ and $r(\delta) \nearrow 2$. Also for all $f \in L_2(0,\infty)$

$$\|D_{s(\delta)} f\|_2 = 2\|f\|_2, \tag{4.5}$$

and for all $f \in L_q(0,\infty)$

$$\|D_{s(\delta)} f\|_q = \delta \|f\|_q. \tag{4.6}$$

Let $\{\Omega_{i,j}\}_{i,j=1}^\infty$ be a partition of $(0,\infty)$ into disjoint measurable sets of infinite measure. For each i, j, let $\varphi_{i,j} : (0,\infty) \to \Omega_{i,j}$ be a one to one onto measure preserving transformation. Let $\delta_i \searrow 0$ and put $s_i = s_i(\delta_i)$, $r_i = r_i(\delta_i)$. Define $f_{i,j} : \Omega_{i,j} \to \mathbb{R}^+$ by,

$$f_{i,j}(\varphi_{i,j}^{-1}(t)) = t^{-1/r_i}, \quad t \in (0,\infty)$$

and $D_{i,j} : M_q(\Omega_{i,j}) \to M_q(\Omega_{i,j})$ by,

$$D_{i,j} f(\varphi_{i,j}(t)) = s_i^{1/r_i} f(\varphi_{i,j}^{-1}(s_i t)).$$

Define also $D : M_q \to M_q$ by $D_{|M_q(\Omega_{i,j})} = D_{i,j}$. Then (denoting by $f_{i,j}$ also the function which is equal to $f_{i,j}$ on $\Omega_{i,j}$ and zero elsewhere),

$$D(f_{i,j}) = f_{i,j}. \tag{4.7}$$

In particular, for each i, D is the identity on the span of $\{f_{i,j}\}_{j=1}^\infty$ which is isometric to ℓ_{r_i}. For all $f \in L_2(0,\infty)$

$$\|Df\|_2 = 2\|f\|_2, \tag{4.8}$$

and for all $f \in L_q(\bigcup_{i=i_0, j=1}^{\infty\ \ \infty} \Omega_{i,j})$

$$\|Df\|_q \le \delta_{i_0} \|f\|_q. \tag{4.9}$$

Note that (4.8) and (4.9) imply that D is bounded (by 2) on M_q.

Let $\varepsilon_{i,j}$, $i, j = 1, 2, \dots$, be an arbitrary sequence of positive numbers and for each i, j let $A_1^{i,j}, \dots, A_{n_{i,j}}^{i,j}$ be a sequence of disjoint sets in $\Omega_{i,j}$ such that the distance of $f_{i,j}$ from the span of $\{\mathbf{1}_{A_k^{i,j}}\}_{k=1}^{n_{i,j}}$ is at most $\varepsilon_{i,j}$, $i, j = 1, 2, \dots$.

Let $m_i \in \mathbb{N}$ be such that $m_i^{\frac{1}{r_i}-\frac{1}{2}} \nearrow \infty$ (recall that $m_i^{\frac{1}{r_i}-\frac{1}{2}}$ is the Banach–Mazur distance of $\ell_{r_i}^{m_1}$ to an Euclidean space) and pick the $\varepsilon_{i,j}$-s to be such that for each i the span of $\{\mathbf{1}_{A_k^{i,j}}\}_{k=1, j=1}^{n_{i,j}\ m_i}$ contains a sequence $\{g_{i,j}\}_{j=1}^{m_i}$ which is a, say, $1/4$-perturbation of $\{f_{i,j}\}_{j=1}^{m_i}$:

$$\|\sum_{i=1}^{m_i} a_j f_{i,j} - \sum_{i=1}^{m_i} a_j g_{i,j}\|_{M_q} < \frac{1}{4}(\sum_{j=1}^{m_i} |a_j|^{r_i})^{1/r_i} \tag{4.10}$$

for all $\{a_j\}_{j=1}^{m_i}$. The properties of D then assure that it preserves a 2-isomorph of $\ell_{r_i}^{m_i}$ up to constant 3.

The space $X = X_q$ we'll use the criterion of Theorem 3.1 on is the span, in M_q, $1 \le q < 2$, of $\{\mathbf{1}_{A_k^{i,j}}\}_{k=1,j=1,i=1}^{n_{i,j}\ m_i\ \infty}$ with

$$x_{i,j,k} = \mathbf{1}_{A_k^{i,j}}/\|\mathbf{1}_{A_k^{i,j}}\|_{M_q}, \quad i = 1,2,\ldots,\ j = 1,\ldots,m_i,\ k = 1,\ldots,n_{i,j},$$

as its 1-unconditional basis. (We used the notation X_q in the beginning of this section for seemingly a different spaces. The two spaces are actually isomorphic, a fact we'll not use it here.) It is easy to see that X_q is complemented in M_q. Actually, the conditional expectation - replacing the values of a function f by their averaged values on each of the sets $A_k^{i,j}$ - is a norm one projection.

The operator T we would like to use is basically D defined above, restricted to X_q. There is a slight problem here as D does not map X_q back to X_q. This is easy to rectify. Note first that for each $\varepsilon > 0$ the sum of the measures of the sets in $\{A_k^{i,j}\}_{k=1,j=1,i=1}^{n_{i,j}\ m_i\ \infty}$ which are of measure smaller than ε is infinite. Otherwise, as is easily verified, $\{x_{i,j,k}\}$ is equivalent in some order to the natural basis of ℓ_q, ℓ_2, or $\ell_q \oplus \ell_2$. But none of these three bases contains block bases 3-equivalent to the natural basis of $\ell_{r_i}^{m_i}$ with $m_i^{\frac{1}{r_i}-\frac{1}{2}} \nearrow \infty$. Now, $D\mathbf{1}_{A_k^{i,j}}$, $i = 1,2,\ldots,\ j = 1,\ldots,m_i,\ k = 1,\ldots,n_{i,j}$ are disjoint characteristic functions, $\mathbf{1}_{B_k^{i,j}}$, $i = 1,2,\ldots,\ j = 1,\ldots,m_i,\ k = 1,\ldots,n_{i,j}$. By the property of the $A_k^{i,j}$-s each of $B_{k'}^{i',j'}$ is equal in distribution to a disjoint union of sets from $\{A_k^{i,j}\}_{k=1,j=1,i=1}^{n_{i,j}\ m_i\ \infty}$ and one can choose the sets in such a manner that each set in $\{A_k^{i,j}\}_{k=1,j=1,i=1}^{n_{i,j}\ m_i\ \infty}$ appears at most once in these representations. It follows that DX is isometric to a subspace of X. So the operator T we'll use is D restricted to X, followed by this isometry. By the sentence following (4.10), T satisfies (a) of Theorem 3.1.

The fact that T satisfies (b) follows from (4.8) and (4.9). Indeed, let E be an m-dimensional subspace of $M_q(\bigcup_{i=i_0,j=1}^{\infty\ \ \infty} \Omega_{i,j})$. (The subspace of M_q of all functions supported on $\bigcup_{i=i_0,j=1}^{\infty\ \ \infty} \Omega_{i,j}$.) It is enough to show that if δ_0 is small enough (depending only on m) then D restricted to E has γ_2 norm at most 6. This will clearly imply that T which is basically the restriction of D to X_q satisfies (b).

The dual of $M_q(\bigcup_{i=i_0,j=1}^{\infty\ \ \infty} \Omega_{i,j})$ is $M_p(\bigcup_{i=i_0,j=1}^{\infty\ \ \infty} \Omega_{i,j})$, $\frac{1}{q} + \frac{1}{p} = 1$. So there is a subspace F of $M_p(\bigcup_{i=i_0,j=1}^{\infty\ \ \infty} \Omega_{i,j})$ of dimension $k(m)$ depending only on m which 2-norms E. Simple duality properties of the γ_2 norm imply that it is enough to prove that if δ_0 is small enough (depending only on m) then D^* restricted to F has γ_2 norm at most 3.

Now M_p is naturally a subspace of $L_p(0,\infty) \oplus_\infty L_2(0,\infty)$ and $D^* : M_p \to M_p$ is the restriction of the operator $K : L_p(0,\infty) \oplus_\infty L_2(0,\infty) \to L_p(0,\infty) \oplus_\infty L_2(0,\infty)$ given by

$$K(f,g) = (D^*f, D^*g).$$

We'll denote by P_1 and P_2 the natural projections onto the first and second components of $L_p(0,\infty) \oplus_\infty L_2(0,\infty)$. (4.8) and (4.9) imply that for all $f \in L_2(0,\infty)$

$$\|D^*f\|_2 = 2\|f\|_2, \tag{4.11}$$

and for all $f \in L_p(\bigcup_{i=i_0,j=1}^{\infty\ \ \infty} \Omega_{i,j})$

$$\|D^*f\|_p \le \delta_{i_0}\|f\|_p. \tag{4.12}$$

The standard inequality, $\gamma_2(S) \leq \|S\|k^{1/2}$ for any operator of rank k, implies that if $\delta_0 < k(m)^{-1/2}$ then the γ_2 norm of KP_2 restricted to F is smaller than 1. Since KP_2 has γ_2 norm 2, we get that $\gamma_2(K_{|F}) < 3$. This implies (using another simple property of the γ_2 norm) that $\gamma_2(D^*_{|F}) < 3$.

The discussion above, Corollary 3.2 and Remark 3.3 imply the main theorem below for $1 < p < 2$. The case $2 < p < \infty$ follows by duality.

Theorem 4.1 (JS). *For $1 < p < \infty, p \neq 2$, $L(L_p[0,1])$ has exactly $2^{\mathfrak{c}}$ different closed ideals.*

Remark 4.2. The proof also gives that M_1 (which is not isomorphic to an L_1 space) has exactly $2^{\mathfrak{c}}$ different closed ideals. By duality also M_∞ has at least $2^{\mathfrak{c}}$ different closed ideals.

Remark 4.3. By Corollary 6.7 below, Theorem 4.1 and Remark 4.2 can be strengthened by interpreting the word “different" to mean mutually non isomorphic as Banach algebras. That is, no two of the ideals admit an homomorphism between them which is continuous in both directions.

5. The non reflexive classical spaces

Here we deal mostly with the number of closed ideals in $L(L_1[0,1])$. The result is less impressive than the one in the previous section as we only prove the existence of a continuum of such ideals. On the other hand the leap from previous results may seem larger, as compared with the case of $L_p[0,1]$, $p > 1$, as previous to **[13]** only a finite number of such ideals were known. We also deal here with the spaces $L(C(0,1))$ and $L(L_\infty[0,1])$.

The result is:

Theorem 5.1 (JPS). *There exists a family $\{\mathcal{I}_p;\ 2 < p < \infty\}$ of (non-closed) ideals in $L(L_1[0,1])$ such that their closures $\overline{\mathcal{I}_p}$ are distinct ideals in $L(L_1[0,1])$. The spaces $L(C(0,1))$ and $L(L_\infty[0,1])$ also have a continuum of closed ideals.*

Remark 5.2. As with the case of $L(L_p[0,1])$, by Corollary 6.7 below, Theorem 5.1 can be strengthened by interpreting the word “distinct" to mean mutually non isomorphic as Banach algebras. That is, no two of the ideals admit an homomorphism between them which is continuous in both directions.

We don't have much to add to the actual proof in **[13]**. We'll only sketch the construction and comment on the idea of the proof. The gist of the construction is a simple lemma, Lemma 5.3, which we bring in full and try to explain its relevance.

In the discussion below we replace $L_1[0,1]$ with its isometric copy, $L_1(\mathbb{T})$. Recall that a set of characters on the circle group, $\mathbb{T}$, equipped with the normalized Lebesgue measure, is called a Λ_p set, $2 < p < \infty$, if the L_p norm on the closed linear span of this set of characters is equivalent to the L_2 norm. For each $2 < p < \infty$, we'll build a sequence of characters of the circle group $\{\gamma_j^p\}_{j=1}^\infty$ which form a Λ_p set and is “as dense as possible" in a certain precise way. We then let J_p be the formal identity from ℓ_1 to this set viewed in

$L_1(\mathbb{T})$; i.e., $J_p : \ell_1 \to L_1(\mathbb{T})$, $J_p e_i = \gamma_i^p$. Each of the ideals $\mathcal{I}_p$, $2 < p < \infty$, in the statement of Theorem 5.1 will be the set of all operators which factor through J_p; i.e.,

$$\mathcal{I}_p = \{AJ_pB;\ B : L_1(\mathbb{T}) \to \ell_1,\ A : L_1(\mathbb{T}) \to L_1(\mathbb{T})\}.$$

To show that the closures of the $\mathcal{I}_p$-s are different we show that for $q > p > 2$, $J_q P$ is not in $\mathcal{I}_p$, where P is a norm one projection from $L_1(\mathbb{T})$ onto (an isometric copy of) ℓ_1.

For $1 \le r < \infty$ and $M \in \mathbb{N}$ we denote L_r over a finite set of cardinality M equipped with the normalized counting measure by L_r^M. We recall that for each $p > 2$ there exists a positive C depending only on p, and for each $N \in \mathbb{N}$ there are vectors $\{v_i\}_{i=1}^N$ in $L_p^{N^{p/2}}$ such that

(1) $\|\sum_{i=1}^N a_i v_i\|_p \le C(\sum_{i=1}^N |a_i|^2)^{1/2}$, and

(2) $\min_{1\le i\le N} \|v_i\|_1 \ge 1$.

One can take the v_i-s to be characters in the span of the first $N^{p/2}$ characters (a space isomorphic with constant depending only on p to $L_p^{N^{p/2}}$). This follows from the solution of Bourgain to the Λ_p problem [5]. The existence of the v_i-s also follows from easier and earlier probabilistic construction of [4] which does not yield characters but is good enough for our purposes. The dimension $N^{p/2}$ is best possible, up to constants depending only on p. The next lemma shows this in greater generality.

Lemma 5.3. *Let $1 \le p < q < \infty$, $\{v_1, \ldots, v_N\} \subset L_q(\mathbb{T})$, and let $T : L_1(\mathbb{T}) \to L_1^{N^{\frac{p}{2}}}$ be an operator. Suppose that C and ϵ satisfy*

(1) $\max_{|\epsilon_i|=1} \|\sum_{i=1}^N \epsilon_i v_i\|_q \le CN^{1/2}$, and

(2) $\min_{1\le i\le N} \|Tv_i\|_1 \ge \epsilon$.

Then $\|T\| \ge (\epsilon/C)N^{\frac{q-p}{2q}}$.

The Lemma and the discussion preceding it should be interpreted in the following way: For each $2 < p < \infty$ and N there is a nicely bounded operator $J_p^N : \ell_1^N \to L_1^{N^{p/2}}$. But for $q > p$, J_q^N does not factor well through J_p^N.

The actual operator J_p is built by gluing together infinitely many J_p^N-s for an increasing sequence of N-s. Also, we repeat each block infinitely often to ensure that $\mathcal{I}_p$ is a subspace, a requirement in the definition of an ideal. The discussion in the previous paragraph hints at the proof that, for $q > p$, J_q doesn't factor through J_p. We'll not repeat the actual construction and proof here and refer the interested reader to the original paper. We do reproduce the proof of Lemma 5.3 here as we believe it should be useful elsewhere and we would like to emphasize its relative simplicity.

Proof of Lemma 5.3. Take u_i^* in $L_\infty^{N^{\frac{p}{2}}} = (L_1^{N^{\frac{p}{2}}})^*$ with $|u_i^*| \equiv 1$ so that $\langle u_i^*, Tv_i\rangle = \|Tv_i\|_1 \ge \epsilon$. Then

$$\begin{aligned}
\epsilon N = \sum_{i=1}^N \langle T^* u_i^*, v_i\rangle &:= \frac{1}{2\pi}\int_0^{2\pi} \sum_{i=1}^N (T^*u_i^*)(a) v_i(a)\, da \\
&\le \frac{1}{2\pi}\int_0^{2\pi} \sup_{a\in[0,1]} |\sum_{i=1}^N (T^*u_i^*)(a) v_i(b)|\, db \\
&=: \frac{1}{2\pi}\int_0^{2\pi} \|\sum_{i=1}^N v_i(b) T^* u_i^*\|_{L_\infty[0,1]} \quad db \\
&\le \|T\| \frac{1}{2\pi}\int_0^{2\pi} \|\sum_{i=1}^N v_i(b) u_i^*\|_{L_\infty^{N^{\frac{p}{2}}}}\, db \\
&\le \|T\| N^{\frac{p}{2q}} \frac{1}{2\pi}\int_0^{2\pi} \Big(\int_{[N^{\frac{p}{2}}]} |\sum_{i=1}^N u_i^*(c) v_i(b)|^q\, dc\Big)^{\frac{1}{q}}\, db \\
&\le \|T\| N^{\frac{p}{2q}} \Big(\int_{[N^{\frac{p}{2}}]} \frac{1}{2\pi}\int_0^{2\pi} |\sum_{i=1}^N u_i^*(c) v_i(b)|^q\, db\, dc\Big)^{\frac{1}{q}} \\
&\le C\|T\| N^{\frac{p+q}{2q}}.
\end{aligned}$$

■

The result stated in Theorem 5.1 for $L(C(0,1))$ and $L(L_\infty[0,1])$ are proved by not completely trivial reasoning from the L_1 case. We'll not repeat the arguments here.

6. Remarks and open problems

The main problem left open here is,

Problem 6.1. How many different closed ideals are there in $L(L_1[0,1])$? $L(C(0,1))$? $L(L_\infty[0,1])$?

Another problem concerning ideals in $L(L_1[0,1])$ comes from the fact that the continuum of ideals built in [13] and discussed in Section 5 are all small.

Problem 6.2. Are there infinitely many large ideals in $L(L_1[0,1])$?

This of course is very much connected with the question of what are the complemented subspaces of $L_1[0,1]$. We repeat the well known simplest question in this direction here.

Problem 6.3. Are there infinite dimensional complemented subspaces of $L_1[0,1]$ which are not isomorphic to either ℓ_1 or $L_1[0,1]$?

Remark 6.4. The ideals constructed in Section 4, based on Corollary 3.2, turn out to be all large. In [13] we also build $2^{\mathfrak{c}}$ small ideals in $L(L_p[0,1])$, $1 < p < \infty$, $p \ne 2$.

Remark 6.5.

1. One can strengthen the conclusion of Corollary 3.2 by getting an antichain of $2^{\mathfrak{c}}$ closed

ideals in $L(X)$; i.e., a collection of $2^{\mathfrak{c}}$ closed ideals no two of which are included one in the other. For that, one just uses a collection of $2^{\mathfrak{c}}$ subsets of C no two of which are included one in the other.
2. Similarly, one gets a collection of $\mathfrak{c}$ different closed ideals in $L(X)$ that form a chain (by taking a chain of subsets of C of that cardinality). It is also easy to show by a density argument that, for any separable X, this is the maximal cardinality of any chain of closed ideals in $L(X)$.
3. Consequently, $L(L_p[0,1])$, $1 < p < \infty$, $p \neq 2$ contains an antichain of cardinality $2^{\mathfrak{c}}$ of closed ideals. It also contains a chain of length $\mathfrak{c}$ of different closed ideals.
4. The construction surveyed in Section 5 also produces a chain of length $\mathfrak{c}$ of closed ideals in $L(L_1[0,1])$.

Next we would like to discuss a stronger notion of distinction between closed ideals (and Banach algebras in general). We say that two Banach algebras A and B are *isomorphically homomorphic* if there is an injective and surjective homomorphism from A onto B which is continuous in both directions. In the literature on Banach algebra the isomorphism of Banach algebras is sometimes understood to be an isometry, i.e., to preserve the norm. We use the ad hoc term *isomorphic homomorphism* to emphasize that we only require the homomorphism to be bounded (equivalently, continuous) in both directions. One could ask,

Question 6.6. Let $1 \leq p < \infty$, $p \neq 2$. How many closed ideals are there in $L(L_p[0,1])$ each two of which are not isomorphically homomorphic?

Eidelheit [8] proved that if X and Y are Banach spaces such that $L(X)$ and $L(Y)$ are isomorphically homomorphic then X and Y are isomorphic Banach spaces. It follows that the $\aleph_1$ ideals in $L(L_p[0,1])$, $1 < p < \infty$, $p \neq 2$ coming from non mutually isomorphic complemented subspaces of $L_p[0,1]$ are mutually non isomorphically homomorphic. Going a bit deeper into the proof of [8] Johnson, Phillips and the author showed that if $\mathcal{I}$ and $\mathcal{J}$ are two closed ideals in $L(X)$ which are isomorphically homomorphic then $\mathcal{I} = \mathcal{J}$. The proof will appear elsewhere. This together with Theorems 5.1, 4.1 and the results of [9], [10] give,

Corollary 6.7.
1. $L(L_1[01,])$ *contains a continuum of mutually non isomorphically homomorphic closed ideals.*
2. For $1 < p < \infty$, $p \neq 2$, $L(L_p[0,1])$ *contains exactly* $2^{\mathfrak{c}}$ *mutually non isomorphically homomorphic closed ideals.*
3. Each of the spaces $L(\ell_p \oplus \ell_q)$, $1 < p < q < \infty$, $L(\ell_p \oplus c_0)$, $L(\ell_p \oplus \ell_\infty)$, $L(\ell_p \oplus \ell_1)$, $1 < p < \infty$ *contains* $2^{\mathfrak{c}}$ *different closed ideals.*

We don't know the answer to the relevant question in the Banach space category:

Problem 6.8. Let $1 \leq p < \infty$, $p \neq 2$. How many closed ideals are there in $L(L_p[0,1])$ each two of which are not isomorphic as Banach spaces?

A result of Arias and Farmer [2] states that for every infinite dimensional complemented subspace X of $L_p[0,1]$, $1 < p < \infty$, which is not isomorphic to a Hilbert space, $L(X)$ is isomorphic (as a Banach space) to $L(L_p[0,1])$. So all the ideals coming from complemented subspaces of $L_p[0,1]$ are isomorphic.

Next we repeat the main problem concerning complemented subspaces of $L_p[0,1]$, $1 < p < \infty$.

Problem 6.9. Is there a continuum of complemented subspaces of $L_p[0,1]$, $1 < p < \infty$, $p \neq 2$, which are mutually non isomorphic?

There was very little progress on new constructions of complemented subspaces of $L_p[0,1]$, $1 < p < \infty$, $p \neq 2$, since [6]. [6] contains a list of still open problems. We would like to repeat one of them as it may appeal to the Harmonic Analysis community. The mutually non-isomorphic $\aleph_1$ complemented subspaces of $L_p[0,1]$ constructed in [6] are all translation invariant subspaces of L_p over the Cantor group $\{-1,1\}^{\mathbb{N}}$ endowed with the natural product measure (which is isometric to $L_p[0,1]$). The projections onto them are translation invariant operators (it is easy to prove that if there is a bounded projection onto a translation invariant subspace then the translation invariant one is also bounded); i.e., idempotent multipliers in $L_p(\{-1,1\}^{\mathbb{N}})$. This produces $\aleph_1$ quite non trivial multipliers. Pełczynski asked whether a similar phenomenon happens on other groups, in particular on $\mathbb{T}$.

Problem 6.10. Are there uncountably many mutually non-isomorphic complemented translation invariant subspaces of $L_p(\mathbb{T})$?

Funding

This work was partially supported by the Israel Science Foundation.

References

[1] S. Argyros, R.G. Haydon, A hereditarily indecomposable L_∞-space that solves the scalar-plus-compact problem. *Acta Math.* **206** (2011), no. 1, 1–54.

[2] A. Arias, J. Farmer, On the structure of tensor products of lp-spaces. *Pacific J. Math.* **175** (1996), no. 1, 13–37.

[3] K. Beanland, T. Kania, N.J. Laustsen, Closed ideals of operators on the Tsirelson and Schreier spaces. *J. Funct. Anal.* **279** (2020), no. 8, 108668, 28 pp.

[4] G. Bennett, L.E. Dor, V. Goodman, W.B. Johnson, C.M. Newman, On uncomplemented subspaces of L_p, $1 < p < 2$. *Israel J. Math.* **26** (1977), no. 2, 178–187.

[5] J. Bourgain, Bounded orthogonal systems and the $\Lambda(p)$–set problem. *Acta Math.* **162** (1989), no. 3-4, 227–245.

[6] J. Bourgain, H.P. Rosenthal, G. Schechtman, An ordinal L^p-index for Banach spaces, with application to complemented subspaces of L^p. *Ann. of Math. (2)* **114** (1981), no. 2, 193–228.

[7] J.W. Calkin, Two-sided ideals and congruences in the ring of bounded operators in Hilbert space. *Ann. of Math. (2)* **42** (1941), 839–873.

[8] M. Eidelheit, On isomorphisms of rings of linear operators. *Studia Math.* **9** (1940), 97–105.

[9] D. Freeman, T. Schlumprecht, A Zsák, The cardinality of the sublattice of closed ideals of operators between certain classical sequence spaces. arXiv:2006.02421

[10] D. Freeman, T. Schlumprecht, A Zsák, Banach spaces for which the space of operators has $2^{\mathfrak{c}}$ closed ideals. arXiv:2006.15415

[11] I.C. Gohberg, A.S. Markus, I.A. Feldman, Normally solvable operators and ideals associated with them. (Russian) *Bul. Akad. Štiince RSS Moldoven.* no. **10(76)**, (1960), 51–70.

[12] W.B. Johnson, B. Maurey, G. Schechtman, L. Tzafriri, Symmetric structures in Banach spaces. *Mem. Amer. Math. Soc.* **19** (1979), no. 217.

[13] W.B. Johnson, G. Pisier, G. Schechtman, Ideals in $L(L_1)$. *Math. Ann.* **376** (2020), no. 1-2, 693–705.

[14] W.B. Johnson, G. Schechtman, Multiplication operators on $L(L_p)$ and ℓ_p-strictly singular operators. *J. Eur. Math. Soc. (JEMS)* **10** (2008), no. 4, 1105–1119.

[15] W.B. Johnson, G. Schechtman, The number of closed ideals in $L(L_p)$. *Acta Math.*, to appear.

[16] S. Kwapień, C. Schütt, Some combinatorial and probabilistic inequalities and their application to Banach space theory. II. *Studia Math.* **95** (1989), no. 2, 141–154.

[17] A. Pietsch, *Operator ideals.* Translated from German by the author. North-Holland Mathematical Library, 20. North-Holland Publishing Co., Amsterdam-New York, 1980. 451 pp. ISBN: 0-444-85293-X

[18] H.P. Rosenthal, On the subspaces of L_p $(p > 2)$ spanned by sequences of independent random variables. *Israel J. Math.* **8** (1970), 273–303.

[19] G. Schechtman, Examples of $\mathcal{L}_p$ spaces $(1 < p \neq 2 < \infty)$. *Israel J. Math.* **22** (1975), no. 2, 138–147.

[20] T. Schlumprecht, A. Zsák, The algebra of bounded linear operators on $\ell_p \oplus \ell_q$ has infinitely many closed ideals. *J. Reine Angew. Math.* **735** (2018), 225–247.

Gideon Schechtman

Departmant of Mathematics, Weizmann Institute of Science, Rehovot, ISRAEL 76100, gideon@weizmann.ac.il