A Schauder basis for $L_1(0, \infty)$ consisting of non-negative functions

William B. Johnson† and Gideon Schechtman‡

Abstract

We construct a Schauder basis for $L_1$ consisting of non-negative functions and investigate unconditionally basic and quasibasic sequences of non-negative functions in $L_p$, $1 \leq p < \infty$.

1 Introduction

In [5], Powell and Spaeth investigate non-negative sequences of functions in $L_p$, $1 \leq p < \infty$, that satisfy some kind of basis condition, with a view to determining whether such a sequence can span all of $L_p$. They prove, for example, that there is no unconditional basis or even unconditional quasibasis (frame) for $L_p$ consisting of non-negative functions. On the other hand, they prove that there are non-negative quasibases and non-negative $M$-bases for $L_p$. The most important question left open by their investigation is whether there is a (Schauder) basis for $L_p$ consisting of non-negative functions. In section 2 we show that there is basis for $L_1$ consisting of non-negative functions.

In section 3 we discuss the structure of unconditionally basic non-negative normalized sequences in $L_p$, $1 \leq p < \infty$. The main result is that such a sequence is equivalent to the unit vector basis of $\ell_p$. We also prove that the

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closed span in $L_p$ of any unconditional quasibasic sequence embeds isomorphically into $\ell_p$.

We use standard Banach space theory, as can be found in [4] or [1]. Let us just mention that $L_p$ is $L_p(0, \infty)$, but inasmuch as this space is isometrically isomorphic under an order preserving operator to $L_p(\mu)$ for any separable purely non-atomic measure $\mu$, our choice of $L(0, \infty)$ rather than e.g. $L_p(0, 1)$ is a matter of convenience. Again as a matter of convenience, in the last part of Section 3 we revert to using $L_p(0, 1)$ as a model for $L_p$.

2 A Schauder basis for $L_1(0, \infty)$ consisting of non-negative functions

For $j = 1, 2, \ldots$ let $\{h^j_{n,i}\}_{n=0,i=1}^{2^n}$ be the mean zero $L_1$ normalized Haar functions on the interval $(j-1, j)$. That is, for $n = 0, 1, \ldots, i = 1, 2, \ldots, 2^n$,

$$h^j_{n,i}(t) = \begin{cases} 2^n j - 1 + \frac{2i-2}{2n+1} & < t < j - 1 + \frac{2i-1}{2n+1} \\ -2^n j - 1 + \frac{2i}{2n+1} & < t < j - 1 + \frac{2i}{2n+1} \\ 0 & \text{otherwise} \end{cases}$$

The system $\{h^j_{n,i}\}_{n=0,i=1}^{2^n}$, in any order which preserves the lexicographic order of $\{h^j_{n,i}\}_{n=0,i=1}^{2^n}$ for each $j$, constitutes a basis for the subspace of $L_1(0, \infty)$ consisting of all functions whose restriction to each interval $(j-1, j)$ have mean zero. To simplify notation, for each $j$ we shall denote by $\{h^j_i\}_{i=1}^{\infty}$ the system $\{h^j_{n,i}\}_{n=0,i=1}^{2^n}$ in its lexicographic order. We shall also denote by $\{h_i\}_{i=1}^{\infty}$ the union of the systems $\{h^j_i\}_{i=1}^{\infty}$, $j = 1, 2, \ldots$, in any order that respects the individual orders of each of the $\{h^j_i\}_{i=1}^{\infty}$.

Let $\pi$ be any permutation of the natural numbers and for each $i \in \mathbb{N}$ let $F_i$ be the two dimensional space spanned by $21(\pi(i) - 1, \pi(i)) + |h_i|$ and $h_i$.

**Proposition 1** $\sum_{i=1}^{\infty} F_i$ is an FDD of $\text{span}^{L_1} \{F_i\}_{i=1}^{\infty}$.

**Proof:** The assertion will follow from the following inequality, which holds for all scalars $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$,

$$\frac{1}{2} \sum_{i=1}^{\infty} |a_i| + \frac{1}{8} \| \sum_{i=1}^{\infty} b_i h_i \| \leq \| \sum_{i=1}^{\infty} a_i (21(\pi(i) - 1, \pi(i)) + |h_i|) + \sum_{i=1}^{\infty} b_i h_i \| \leq 3 \sum_{i=1}^{\infty} |a_i| + \| \sum_{i=1}^{\infty} b_i h_i \|.$$  \hspace{1cm} (1)
The right inequality in (1) follows easily from the triangle inequality. As for the left inequality, notice that the conditional expectation projection onto the closed span of \( \{1_{(i-1,i)}\}_{i=1}^{\infty} \) is of norm one and the complementary projection, onto the closed span of \( \{h_i\}_{i=1}^{\infty} \), is of norm 2. It follows that

\[
\| \sum_{i=1}^{\infty} a_i (21_{(\pi(i)-1,\pi(i))}) + \sum_{i=1}^{\infty} b_i h_i \| \geq \max \{ 2 \sum_{i=1}^{\infty} |a_i|, \frac{1}{2} \sum_{i=1}^{\infty} b_i h_i \}.
\]

Since \( \sum_{i=1}^{\infty} a_i |h_i| \leq \sum_{i=1}^{\infty} |a_i| \), we get

\[
\| \sum_{i=1}^{\infty} a_i (21_{(\pi(i)-1,\pi(i))}) + |h_i| + \sum_{i=1}^{\infty} b_i h_i \| \geq \max \{ \sum_{i=1}^{\infty} |a_i|, \frac{1}{4} \sum_{i=1}^{\infty} b_i h_i \}
\]

from which the left hand side inequality in (1) follows easily. \( \square \)

**Proposition 2** Let \( \pi \) be any permutation of the natural numbers and for each \( i \in \mathbb{N} \) let \( F_i \) be the two dimensional space spanned by \( 21_{(\pi(i)-1,\pi(i))} + |h_i| \) and \( h_i \). Then \( \text{span} \{ F_i \}_{i=1}^{\infty} \) admits a basis consisting of non-negative functions.

**Proof:** In view of Proposition 1 it is enough to show that each \( F_i \) has a two term basis consisting of non-negative functions and with uniform basis constant. Put \( x_i = 21_{(\pi(i)-1,\pi(i))} + |h_i| + h_i \) and \( y_i = 21_{(\pi(i)-1,\pi(i))} + |h_i| - h_i \). Then clearly \( x_i, y_i \geq 0 \) everywhere and \( \|x_i\| = \|y_i\| = 3 \). We now distinguish two cases: If \( 1_{(\pi(i)-1,\pi(i))} \) is disjoint from the support of \( h_i \) then, for all scalars \( a, b \),

\[
\|ax_i + by_i\| \geq \|a(|h_i| + h_i) + b(|h_i| - h_i)\| = |a| + |b|.
\]

If the support of \( h_i \) is included in \( (\pi(i) - 1, \pi(i)) \), Let \( 2^{-s} \) be the size of that support, \( s \geq 0 \). Then for all scalars \( a, b \),

\[
\|ax_i + by_i\| \geq \|a(|h_i| + h_i) + b(|h_i| - h_i) + 2(a + b)1_{\text{supp}(h_i)}\|
\]

\[
= 2^{-s-1}(|(2s+1)^2a + 2b| + (2s+1)^2b + 2a) \geq \max\{|a|, |b|\}.
\]

**Theorem 1** \( L_1(0, \infty) \), and consequently any separable \( L_1 \) space, admits a Schauder basis consisting of non-negative functions.
Proof: When choosing the order on \( \{h_i\} \) we can and shall assume that \( h_1 = h_{0,1}^1; \) i.e., the first mean zero Haar function on the interval \((0,1)\). Let \( \pi \) be any permutation of \( \mathbb{N} \) such that \( \pi(1) = 1 \) and for \( i > 1 \), if \( h_i = h_{n,k}^i \) for some \( n, k, \) and \( j \) then \( \pi(i) > j \). It follows that except for \( i = 1 \) the support of \( h_i \) is disjoint from the interval \((\pi(i) - 1, \pi(i))\). It is easy to see that such a permutation exists. We shall show that under these assumptions \( \sum_{i=1}^{\infty} F_i \)
spans \( L_1(0, \infty) \) and, in view of Proposition 2, this will prove the theorem for \( L_1(0, \infty) \). First, since \( \pi(1) = 1 \) we get that \( 31_{(0,1)} = 21_{(\pi(1)-1,\pi(1))} + |h_1| \in F_1 \), and since all the mean zero Haar functions on \((0,1)\) are clearly in \( \sum_{i=1}^{\infty} F_i \), we get that \( L_1(0,1) \subset \sum_{i=1}^{\infty} F_i \).

Assume by induction that \( L_1(0,j) \subset \sum_{i=1}^{\infty} F_i \). Let \( l \) be such that \( \pi(l) = j+1 \). By our assumption on \( \pi \), the support of \( h_l \) is included in \((0,j)\), and so by the induction hypothesis, \( |h_l| \in \sum_{i=1}^{\infty} F_i \). Since also \( 21_{(j,j+1)} + |h_l| \in \sum_{i=1}^{\infty} F_i \) we get that \( 1_{(j,j+1)} \in \sum_{i=1}^{\infty} F_i \). Since the mean zero Haar functions on \((j,j+1)\)
are also in \( \sum_{i=1}^{\infty} F_i \) we conclude that \( L_1(0,j+1) \subset \sum_{i=1}^{\infty} F_i \).

This finishes the proof for \( L_1(0, \infty) \). Since every separable \( L_1 \) space is order isometric to one of the spaces \( \ell_1^k \), \( k = 1, 2, \ldots, \ell_1 \), \( L_1(0, \infty), L_1(0, \infty) \bigoplus_1 \ell_1^k \), \( k = 1, 2, \ldots, \) or \( L_1(0, \infty) \bigoplus_1 \ell_1 \), and since the discrete \( L_1 \) spaces \( \ell_1^k \), \( k = 1, 2, \ldots, \) and \( \ell_1 \) clearly have non-negative bases, we get the conclusion for any separable \( L_1 \) space.

\section{Unconditional non-negative sequences in \( L_p \)}

Here we prove

**Theorem 2** Suppose that \( \{x_n\}_{n=1}^{\infty} \) is a normalized unconditionally basic sequence of non-negative functions in \( L_p \), \( 1 \leq p < \infty \). Then \( \{x_n\}_{n=1}^{\infty} \) is equivalent to the unit vector basis of \( \ell_p \).

**Proof:** First we give a sketch of the proof, which should be enough for experts in Banach space theory. By unconditionality, we have for all coefficients \( a_n \) that \( \| \sum_n a_n x_n \|_p \) is equivalent to the square function \( \|(\sum_n |a_n|^2 x_n^2)^{1/2}\|_p \), and, by non-negativity of \( x_n \), is also equivalent to \( \| \sum_n |a_n|x_n|_p \). Thus by trivial interpolation when \( 1 \leq p \leq 2 \), and by extrapolation when \( 2 < p < \infty \), we see that \( \| \sum_n a_n x_n \|_p \) is equivalent to \( \| (\sum_n |a_n|^p x_n^p)^{1/p} \|_p = (\sum_n |a_n|^p)^{1/p} \).

We now give a formal argument for the benefit of readers who are not familiar with the background we assumed when giving the sketch. Let \( K \) be
the unconditional constant of \(\{x_n\}_{n=1}^{\infty}\). Then

\[
K^{-1}\|\sum_{n=1}^{N} a_n x_n\|_p \leq B_p \|\left(\sum_{n=1}^{N} |a_n|^2 x_n^2\right)^{1/2}\|_p
\]

\[
\leq B_p \|\sum_{n=1}^{N} |a_n| x_n\|_p \leq B_p K \|\sum_{n=1}^{N} a_n x_n\|_p,
\]

where the first inequality is obtained by integrating against the Rademacher functions (see, e.g., [4, Theorem 2.b.3]). The constant \(B_p\) is Khintchine’s constant, so \(B_p = 1\) for \(p \leq 2\) and \(B_p\) is of order \(\sqrt{p}\) for \(p > 2\). If \(1 \leq p \leq 2\) we get from (2)

\[
K^{-1}\|\sum_{n=1}^{N} a_n x_n\|_p \leq \|\left(\sum_{n=1}^{N} |a_n|^p x_n^p\right)^{1/p}\|_p \leq K \|\sum_{n=1}^{N} a_n x_n\|_p.
\]

Since \(\|\sum_{n=1}^{N} |a_n|^p x_n^p\|^{1/p}_p = \left(\sum_{n=1}^{N} |a_n|^p\right)^{1/p}\), this completes the proof when \(1 \leq p \leq 2\). When \(2 < p < \infty\), we need to extrapolate rather than do (trivial) interpolation. Write \(1/2 = \theta/1 + (1 - \theta)/p\). Then

\[
(KB_p)^{-1}\|\sum_{n=1}^{N} a_n x_n\|_p \leq \|\left(\sum_{n=1}^{N} |a_n|^2 x_n^2\right)^{1/2}\|_p
\]

\[
\leq \|\sum_{n=1}^{N} |a_n| x_n\|^\theta_1 \|\left(\sum_{n=1}^{N} |a_n|^p x_n^p\right)^{1/p}\|^{1-\theta}_p
\]

\[
\leq K \|\sum_{n=1}^{N} a_n x_n\|^\theta_1 \|\left(\sum_{n=1}^{N} |a_n|^p\right)^{(1-\theta)/p}\),
\]

so that

\[
(K^2B_p)^{(1-\theta)/(1-\theta)}\|\sum_{n=1}^{N} a_n x_n\|_p \leq \left(\sum_{n=1}^{N} |a_n|^p\right)^{1/p} \leq K \|\sum_{n=1}^{N} a_n x_n\|_p.
\]

As stated, Theorem 2 gives no information when \(p = 2\) because every normalized unconditionally basic sequence in a Hilbert space is equivalent to the unit vector basis of \(\ell_2\). However, if we extrapolate slightly differently in the above argument (writing \(1/2 = \theta/1 + (1 - \theta)/\infty\)) we see that, no matter what \(p\) is, \(\|\sum_{n=1}^{N} a_n x_n\|_p\) is also equivalent to \(\|\max_n |a_n| x_n\|_p\). From this one can deduce e.g. that only finitely many Rademachers can be in the closed
span of \( \{x_n\}_{n=1}^\infty \); in particular, \( \{x_n\}_{n=1}^\infty \) cannot be a basis for \( L_p \) even when \( p = 2 \). However, the proof given in [5] that a normalized unconditionally basic sequence of non-negative functions \( \{x_n\}_{n=1}^\infty \) in \( L_p \) cannot span \( L_p \) actually shows that only finitely many Rademachers can be in the closed span of \( \{x_n\}_{n=1}^\infty \). This is improved in our last result, which shows that the closed span of an unconditionally non-negative quasibasic sequence in \( L_p(0,1) \) cannot contain any strongly embedded infinite dimensional subspace (a subspace \( X \) of \( L_p(0,1) \) is said to be strongly embedded if the \( L_p(0,1) \) norm is equivalent to the \( L_r(0,1) \) norm on \( X \) for some – or, equivalently, for all – \( r < p \); see e.g. [1, p. 151]). The main work for proving this is contained in Lemma 1.

Before stating Lemma 1, we recall that a quasibasis for a Banach space \( X \) is a sequence \( \{f_n, g_n\}_{n=1}^\infty \) in \( X \times X^* \) such that for each \( x \) in \( X \) the series \( \sum_n \langle g_n, x \rangle f_n \) converges to \( x \). (In [5] a sequence \( \{f_n\}_{n=1}^\infty \) in \( X \) is a called a quasibasis for \( X \) provided there exists such a sequence \( \{g_n\}_{n=1}^\infty \). Since the sequence \( \{g_n\}_{n=1}^\infty \) is typically not unique, we prefer to specify it up front.) The quasibasis \( \{f_n, g_n\}_{n=1}^\infty \) is said to be unconditional provided that for each \( x \) in \( X \) the series \( \sum_n \langle g_n, x \rangle f_n \) converges unconditionally to \( x \). One then gets from the uniform boundedness principle (see, e.g., [5, Lemma 3.2]) that there is a constant \( K \) so that for all \( x \) and all scalars \( a_n \) with \( |a_n| \leq 1 \), we have \( \| \sum_n a_n \langle g_n, x \rangle f_n \| \leq K \| x \| \). A sequence \( \{f_n, g_n\}_{n=1}^\infty \) in \( X \times X^* \) is said to be [unconditionally] quasibasic provided \( \{f_n, h_n\}_{n=1}^\infty \) is an [unconditional] quasibasis for the closed span \( \{f_n\} \) of \( \{f_n\}_{n=1}^\infty \), where \( h_n \) is the restriction of \( g_n \) to \( \{f_n\} \).

**Lemma 1** Suppose that \( \{f_n, g_n\}_{n=1}^\infty \) is an unconditionally quasibasic sequence in \( L_p(0,1) \), \( 1 < p < \infty \) with each \( f_n \) non-negative. If \( \{y_n\}_{n=1}^\infty \) is a normalized weakly null sequence in \( \{f_n\} \), then \( \|y_n\|_1 \to 0 \) as \( n \to \infty \).

**Proof:** If the conclusion is false, we get a normalized weakly null sequence \( \{y_n\}_{n=1}^\infty \) in \( \{f_n\} \) and a \( c > 0 \) so that for all \( n \) we have \( \|y_n\|_1 > c \).

By passing to a subsequence of \( \{y_n\}_{n=1}^\infty \), we can assume that there are integers \( 0 = m_1 < m_2 < \ldots \) so that for each \( n \),

\[
\sum_{k=1}^{m_n} |\langle g_k, y_n \rangle| f_k |p < 2^{-n-3}c \quad \text{and} \quad \sum_{k=m_n+1}^{\infty} |\langle g_k, y_n \rangle| f_k |p < 2^{-n-3}c. \tag{5}
\]

Effecting the first inequality in (5) is no problem because \( y_n \to 0 \) weakly, but the second inequality perhaps requires a comment. Once we have a
of $y_n$ that satisfies the first inequality in (5), from the unconditional convergence of the expansion of $y_n$ and the non-negativity of all $f_k$ we get that $\| \sum_{k=N}^{\infty} |\langle g_k, y_n \rangle| f_k \|_p \to 0$ as $n \to \infty$, which allows us to select $m_{n+1}$ to satisfy the second inequality in (5).

Since $\|y_n\|_1 > c$, from (5) we also have for every $n$ that

$$\| \sum_{k=m_n+1}^{m_{n+1}} |\langle g_k, y_n \rangle| f_k \|_1 \geq \| \sum_{k=m_n+1}^{m_{n+1}} \langle g_k, y_n \rangle f_k \|_1 \geq c/2. \quad (6)$$

Since $L_p$ has an unconditional basis, by passing to a further subsequence we can assume that $\{y_n\}_{n=1}^{\infty}$ is unconditionally basic with constant $K_p$. Also, $L_p$ has type $s$, where $s = p \wedge 2$ (see [1, Theorem 6.2.14]), so for some constant $K'_p$ we have for every $N$ the inequality

$$\| \sum_{n=1}^{N} y_n \|_p \leq K'_p N^{1/s}. \quad (7)$$

On the other hand, letting $\delta_k = \text{sign} \langle g_k, y_n \rangle$ when $m_n + 1 \leq k \leq m_{n+1}$, $n = 1, 2, 3, \ldots$, we have

$$KK_p \| \sum_{n=1}^{N} y_n \|_p \geq K_p \| \sum_{n=1}^{N} \sum_{k=m_n+1}^{m_{n+1}} \delta_k \langle g_k, y_n \rangle f_k \|_p$$

$$\geq \| \sum_{n=1}^{N} \sum_{k=m_n+1}^{m_{n+1}} \langle g_k, y_n \rangle f_k \|_p - \| \sum_{n=1}^{N} \sum_{k \not\in [m_n+1,m_{n+1}]} \delta_k \langle g_k, y_n \rangle f_k \|_p$$

$$\geq \| \sum_{n=1}^{N} \sum_{k=m_n+1}^{m_{n+1}} \langle g_k, y_n \rangle f_k \|_1 - \| \sum_{n=1}^{N} \sum_{k \not\in [m_n+1,m_{n+1}]} \langle g_k, y_n \rangle f_k \|_p$$

$$\geq \sum_{n=1}^{N} \| \sum_{k=m_n+1}^{m_{n+1}} \langle g_k, y_n \rangle f_k \|_1$$

$$- \sum_{n=1}^{N} \left( \sum_{k=1}^{m_n} |\langle g_k, y_n \rangle| f_k \|_p + \| \sum_{k=m_n+1+1}^{\infty} |\langle g_k, y_n \rangle| f_k \|_p \right)$$

$$\geq Nc/2 - c/4 \quad \text{by (6) and (5)} \quad (8)$$

This contradicts (7).
Theorem 3 Suppose that \( \{f_n, g_n\}_{n=1}^\infty \) is an unconditional quasibasic sequence in \( L_p(0,1) \), \( 1 \leq p < \infty \), and each \( f_n \) is non-negative. Then the closed span \([f_n]\) of \( \{f_n\}_{n=1}^\infty \) embeds isomorphically into \( \ell_p \).

Proof: The case \( p = 1 \) is especially easy: Assume, as we may, that \( \|f_n\|_1 = 1 \). There is a constant \( K \) so that for each \( y \) in \([f_n]\)

\[
\|y\|_1 \leq \| \sum_{n=1}^\infty |\langle g_n, y \rangle| f_n \|_1 \leq K \|y\|_1,
\]

hence the mapping \( y \mapsto \{\langle g_k, y \rangle\}_{k=1}^\infty \) is an isomorphism from \([f_n]\) into \( \ell_1 \).

So in the sequel assume that \( p > 1 \). From Lemma 1 and standard arguments (see, e.g., [1, Theorem 6.4.7]) we have that every normalized weakly null sequence in \([f_n]\) has a subsequence that is an arbitrarily small perturbation of a disjoint sequence and hence the subsequence is \( 1 + \epsilon \)-equivalent to the unit vector basis for \( \ell_p \). This implies that \([f_n]\) embeds isomorphically into \( \ell_p \) (see [3] for the case \( p > 2 \) and [2, Theorems III.9, III.1, and III.2] for the case \( p < 2 \)).

References


W. B. Johnson
Department of Mathematics
Texas A&M University
College Station, TX 77843 U.S.A.
johnson@math.tamu.edu

G. Schechtman
Department of Mathematics
Weizmann Institute of Science
Rehovot, Israel
gideon@weizmann.ac.il