Entropy versus influence for complex functions of modulus one *

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Abstract

We present an example of a function f from $\{-1,1\}^n$ to the unit sphere in $\mathbb{C}$ with influence bounded by 1 and entropy of $|\hat{f}|^2$ larger than $\frac{1}{2}\log n$. We also present an example of a function f from $\{-1,1\}^n$ to $\mathbb{R}$ with L_2 norm 1, L_∞ norm bounded by $\sqrt{2}$, influence bounded by 1 and entropy of $\hat{f}^2$ larger than $\frac{1}{2}\log n$.

1 Introduction

We denote by $\varepsilon_i : \{-1,1\}^n \to \{-1,1\}$ the projection onto the i-s coordinate: $\varepsilon_i(\delta_1,\dots,\delta_n) = \delta_i$. For a subset A of $[n] := \{1,\dots,n\}$ we denote $W_A = \prod_{i\in A}\varepsilon_i$, $W_A : \{-1,1\}^n \to \{-1,1\}$. The W_A-s are the characters of the Cantor group $\{-1,1\}^n$ (with coordintewise multiplication) and form an orthonormal basis in L_2 of the Cantor group equipped with the normalized counting measure. In most of this note we shall be concerned with functions from $\{-1,1\}^n$ into the real numbers, $\mathbb{R}$, but later on we shall also consider functions into the complex plane, $\mathbb{C}$. These can also be considered as a couple of real functions. Each such function $f : \{-1,1\}^n \to \mathbb{C}$ has a unique expansion

$$f = \sum_{A\subseteq[n]} \hat{f}(A)W_A,$$

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where $\hat{f}(A) \in \mathbb{C}$ are given by

$$\hat{f}(A) = \mathbf{E}(fW_A) = 2^{-n} \sum_{\varepsilon \in \{-1,1\}^n} f(\varepsilon) W_A(\varepsilon).$$

Note that if $f : \{-1,1\}^n \to \mathbb{R}$, then $\hat{f}(A) \in \mathbb{R}$ for every $A \subseteq [n]$. The orthonormality of the W_A-s implies that

$$\|f\|_2^2 := 2^{-n} \sum_{\varepsilon \in \{-1,1\}^n} |f(\varepsilon)|^2 = \sum_{A \subseteq [n]} |\hat{f}(A)|^2.$$

Define the influence of a function $f : \{-1,1\}^n \to \mathbb{C}$ by

$$I(f) = \sum_{A \subseteq [n]} |\hat{f}(A)|^2 |A| \tag{1}$$

where for $A \subseteq [n]$, $|A|$ denotes the cardinality of A. This object, especially for boolean functions, is a deeply studied one and quite influential (but this is not the reason for the name...) in several directions. We refer to [O] for some information. A recent paper dealing with the subject is [KKLMS].

The entropy of the sequence $|\hat{f}(A)|^2$ is given by

$$H(|\hat{f}(A)|^2) = -\sum_{A \subseteq [n]} |\hat{f}(A)|^2 \log |\hat{f}(A)|^2. \tag{2}$$

$(0 \log 0 := 0)$. The base of the log does not really matter here (as long as it is consistent throughout the paper, so the log in the statements of the results is the same as the one here). For concreteness we take the log to base 2. Note that if f has L_2 norm 1 then the sequence $\{|\hat{f}(A)|^2\}_{A \subseteq [n]}$ sums up to 1 and thus this is the usual definition of entropy of this probability distribution, but we shall use this notation and term also for non normalized functions.

The entropy influence conjecture of Friedgut and Kalai [FK] is that for some absolute constant K, for all n and all boolean functions $f : \{-1,1\}^n \to \{-1,1\}$

$$H(|\hat{f}(A)|^2) \leq K I(f).$$

For the significance of this conjecture we refer to the original paper [FK], and to Kalai's blog [K] (embedded in Tao's blog) which report on all significant results concerning the conjecture. [KKLMS] establishes a weaker version of the conjecture. Its introduction is also a good source of information on the problem.

As is well known the conjecture fails if we replace boolean functions with general real functions (say, normalized to have L_2 norm 1): $f(\varepsilon_1,\dots,\varepsilon_n) = n^{-1/2}\sum_{i=1}^n \varepsilon_i$ gives such an example.

In this note we show that the analogous version of the conjecture for two classes of (L_2 normalized) functions which resemble boolean functions fail as well.

The first class is the set of well bounded real functions on $\{-1,1\}^n$. The second is the complex functions on $\{-1,1\}^n$ which have modulus 1. The second example solves a question raised by Gady Kozma some time ago (see [K], comment from April 2, 2011). More specifically, we prove the following two theorems:

Theorem 1. *For each $n = 1, 2, \dots$ there is a function $f : \{-1,1\}^n \to \mathbb{R}$ with*

$$\|f\|_2 = 1 \le \|f\|_\infty \le \sqrt{2}, \quad I(f) < 1, \quad \textit{and} \quad H(\hat{f}^2) > \frac{n}{n+1}\log n.$$

Theorem 2. *For each $n = 1, 2, \dots$ there is a function $f : \{-1,1\}^n \to \{z \in \mathbb{C};\ |z| = 1\}$ with*

$$I(f) < 1, \quad \textit{and} \quad H(|\hat{f}|^2) > \frac{n}{n+1}\log n.$$

Actually, Theorem 1 (with a somewhat different lower bound for $H(|\hat{f}|^2)$ but still of order $\log n$) follows from Theorem 2 but we prefer to give an easy independent proof.

The innovative contribution of this note is in the idea of the examples. The proofs are elementary, easy and self contained. We hope that some variation of the examples will serve related purposes.

After the first version of this note was posted, Joe Neeman sent me a simpler proof of Theorem 1 (with different constants). See the remark at the end of this note. Also, an anonymous referee wrote that theorem 2 was known to experts but not published. No hint was given for the construction.

Consequently, this note will not be further sent for publication (at least in the near future).

2 The main results

Define $P_0 \equiv Q_0 \equiv 1$ and, given a sequence $\{a_n\}_{n=1}^{\infty} \subset (0,1]$, define $P_n, Q_n : \{-1,1\}^n \to \mathbb{R}$ inductively by

$$P_{n+1} = P_n + \varepsilon_{n+1} a_{n+1} Q_n, \quad Q_{n+1} = \varepsilon_{n+1} a_{n+1} P_n - Q_n, \quad n = 0, 1, \dots \tag{3}$$

The definition is inspired by that of the Rudin–Shapiro polynomials. Actually, for the constant sequence $a_n = 1$ for all n, P_n and Q_n are Rudin–Shapiro polynomials. These are functions $R_n : \{-1,1\}^n \to \mathbb{R}$ all having all Fourier–Walsh coefficients of absolute value one (and in particular $\|R_n\|_2 = 2^{n/2}$) and $\|R_n\|_\infty \le \sqrt{2}\|R_n\|_2$. These of course are no good for our purposes as $2^{-n/2}R_n$ has influence equals to $n/2$ and Fourier entropy n.

Proposition 1. *For all $n = 1, 2, \dots$, $|P_n|^2 + |Q_n|^2$ is a constant function,*

$$|P_n|^2 + |Q_n|^2 \equiv 2\prod_{i=1}^{n}(1 + a_i^2), \tag{4}$$

$$\|P_n\|_2 = \|Q_n\|_2 = \prod_{i=1}^{n}(1 + a_i^2)^{1/2}, \tag{5}$$

$$\prod_{i=1}^{n}(1 + a_i^2)^{1/2} \le \|P_n\|_\infty, \|Q_n\|_\infty \le \sqrt{2}\prod_{i=1}^{n}(1 + a_i^2)^{1/2}. \tag{6}$$

For each $A \subseteq [n]$,

$$\hat{P}_n(A)^2 = \hat{Q}_n(A)^2 = \prod_{i \in A} a_i^2. \tag{7}$$

Proof: By Induction, starting from $n = 0$. (4) holds for $n = 0$ (or check for $n = 1$, if the case $n = 0$ bothers you). Assume it holds for n then by (3),

$$|P_{n+1}|^2 + |Q_{n+1}|^2 = (1 + a_{n+1}^2)(|P_n|^2 + |Q_n|^2) \equiv 2\prod_{i=1}^{n+1}(1 + a_i^2).$$

(5) holds for $n = 0$. Assume it holds for n then the orthogonality of P_n and $\varepsilon_{n+1}Q_n$ implies

$$\|P_{n+1}\|_2^2 = \|P_n\|_2^2 + a_{n+1}^2\|Q_n\|_2^2 = \|Q_n\|_2^2 + a_{n+1}^2\|P_n\|_2^2 = \|Q_{n+1}\|_2^2.$$

The second equality in (5) now follows from (4).
(6) follows from (4) and (5).

The last assertion of the Proposition, (7), follows easily from the inductive definition. (The proof of Theorem 2 below contains a proof of a more general assertion.) ■

Proposition 2. *For each* $n = 1, 2, \ldots$

$$I(P_n) = I(Q_n) = \sum_{i=1}^{n} a_i^2 \prod_{j \neq i}^{n} (1 + a_j^2). \tag{8}$$

Proof: Note first that for each two functions $R, S : \{-1, 1\}^n \to \mathbb{R}$ and $a \in \mathbb{R}$, if we put $T = R + a\varepsilon_{n+1}S$, i.e., $T(\varepsilon_1, \ldots, \varepsilon_{n+1}) = R(\varepsilon_1, \ldots, \varepsilon_n) + a\varepsilon_{n+1}S(\varepsilon_1, \ldots, \varepsilon_n)$, then

$$I(T) = I(R) + a^2(I(S) + \|S\|_2^2). \tag{9}$$

We now prove (8) by induction. $I(P_0) = I(Q_0) = 0$ (and $I(P_1) = I(Q_1) = a_1^2$). Assume (8) then by (9) and (5),

$$\begin{aligned} I(P_{n+1}) &= I(P_n) + a_{n+1}^2 (I(Q_n) + \prod_{i=1}^{n} (1 + a_i^2)) \\ &= \sum_{i=1}^{n} a_i^2 \prod_{1 \le j \le n, j \neq i} (1 + a_j^2) + a_{n+1}^2 \sum_{i=1}^{n} a_i^2 \prod_{1 \le j \le n, j \neq i} (1 + a_j^2) + a_{n+1}^2 \prod_{i=1}^{n} (1 + a_i^2) \\ &= \sum_{i=1}^{n} a_i^2 \prod_{1 \le j \le n+1, j \neq i} (1 + a_j^2) + a_{n+1}^2 \prod_{i=1}^{n} (1 + a_i^2) = \sum_{i=1}^{n+1} a_i^2 \prod_{1 \le j \le n+1, j \neq i} (1 + a_j^2). \end{aligned} \tag{10}$$

The proof for $I(Q_{n+1})$ is almost identical. ■

Remark 1. It follows that putting $K = \sum_{i=1}^{n} a_i^2$,

$$I(P_n) = I(Q_n) < Ke^K.$$

Proposition 3. *For each* $n = 1, 2, \ldots$

$$H(\hat{P}_n^2) = H(\hat{Q}_n^2) = -\sum_{i=1}^{n} \left(\prod_{1 \le j \le n, j \neq i} (1 + a_j^2) \right) a_i^2 \log a_i^2. \tag{11}$$

Proof: By the last assertion of Proposition 1

$$
\begin{aligned}
H(\hat{P}_n^2) &= -\sum_{A\subseteq[n]}\left(\prod_{j\in A}a_j^2\right)\log\left(\prod_{i\in A}a_i^2\right)\\
&= -\sum_{A\subseteq[n]}\left(\prod_{j\in A}a_j^2\right)\sum_{i\in A}\log a_i^2 = -\sum_{i=1}^n(\log a_i^2)\sum_{A,i\in A}\prod_{j\in A}a_j^2\\
&= -\sum_{i=1}^n a_i^2(\log a_i^2)\sum_{A,i\in A}\prod_{j\in A\setminus\{i\}}a_j^2 = -\sum_{i=1}^n a_i^2(\log a_i^2)\prod_{j\neq i}(1+a_j^2).
\end{aligned}
$$

■

Theorem 1. *For each $n = 1, 2, \ldots$ there is a function $f : \{-1, 1\}^n \to \mathbb{R}$ with*

$$\|f\|_2 = 1 \leq \|f\|_\infty \leq \sqrt{2}, \quad I(f) < 1, \quad \textit{and} \quad H(\hat{f}^2) > \frac{n}{n+1}\log n.$$

Proof: Note that for any $g : \{-1, 1\}^n \to \mathbb{R}$ and $a \in \mathbb{R}$,

$$I(ag) = a^2 I(g) \quad \text{and} \quad H(\widehat{ag}^2) = a^2 H(\hat{g}^2) - (a^2\log a^2)\|g\|_2^2. \tag{12}$$

Put

$$f = \frac{P_n}{\|P_n\|_2} = \prod_{i=1}^n(1+a_i^2)^{-1/2}P_n.$$

(The last equation is by (5).) Then $\|f\|_2 = 1$ and $\|f\|_\infty \leq \sqrt{2}$ (by (6)). By the rescaling (12), (8) and (11),

$$I(f) = \sum_{i=1}^n \frac{a_i^2}{1+a_i^2},$$

and

$$
\begin{aligned}
H(\hat{f}^2) &= -\sum_{i=1}^n\frac{a_i^2}{1+a_i^2}\log a_i^2 - \log\prod_{i=1}^n(1+a_i^2)^{-1}\\
&> \frac{-1}{1+\max_{1\leq i\leq n}a_i^2}\sum_{i=1}^n a_i^2\log a_i^2.
\end{aligned}
$$

The choice $a_i = 1/\sqrt{n}, i = 1, \ldots, n$, gives

$$I(f) < 1 \quad \text{and} \quad H(\hat{f}^2) > \frac{1}{1+n^{-1}} \log n.$$

∎

Remark 2. The functions produced above have expectations different from zero. It is easy to rectify this by looking instead at $\varepsilon_{n+1} f : \{-1,1\}^{n+1} \to \mathbb{R}$. The function $g = \varepsilon_{n+1} f$ has the same L_2, L_∞ and $H(\hat{g}^2)$ as f and $I(g) = I(f) + 1 < 2$.

We denote by S^1 the unit sphere in $\mathbb{R}^2$: $S^1 = \{z \in \mathbb{C};\ |z| = 1\}$

Theorem 2. *For each $n = 1, 2, \ldots$ there is a function $f : \{-1,1\}^n \to S^1$ with*

$$I(f) < 1, \quad \textit{and} \quad H(|\hat{f}|^2) > \frac{n}{n+1} \log n.$$

Proof: Complex functions satisfy a similar scaling property to (12): For any $g : \{-1,1\}^n \to \mathbb{C}$ and $a \in \mathbb{C}$,

$$I(ag) = |a|^2 I(g) \quad \text{and} \quad H(|\widehat{ag}|^2) = |a|^2 H(|\hat{g}|^2) - (|a|^2 \log |a|^2) \|g\|_2^2. \tag{13}$$

Also, the absolute values of the Fourier–Walsh coefficients are preserved under taking conjugates and in particular,

$$H(|\hat{\bar{g}}|^2) = H(|\hat{g}|^2). \tag{14}$$

Put

$$f_n = \frac{1}{\sqrt{2} \prod_{i=1}^n (1 + a_i^2)^{1/2}} (P_n + \imath Q_n).$$

By (4) $f_n : \{-1,1\} \to S^1$. By (13) and (8),

$$I(f_n) = \sum_{i=1}^n \frac{a_i^2}{1 + a_i^2}. \tag{15}$$

To evaluate $H(|\hat{f}_n|^2)$ notice first that, for each of the four functions $F_n^1 = P_n + \imath Q_n, F_n^2 = P_n - \imath Q_n, F_n^3 = Q_n + \imath P_n$ and $F_n^4 = Q_n - \imath P_n$, each of the four sets $\{|\widehat{F_n^k}(A)|^2\}_{A\subseteq[n]}$, $k = 1,2,3,4$, is equal to $\{2\prod_{i\in A} a_i^2\}_{A\subseteq[n]}$. More precisely, for each $A \subseteq [n]$ and $k = 1,2,3,4$,

$$|\widehat{F_n^k}(A)| = \sqrt{2}\prod_{i\in A}|a_i|.$$

This is easily proved by induction on n: Assuming the assertion for n then by (3),

$$F_{n+1}^1 = P_n - \imath Q_n + \varepsilon_{n+1}a_{n+1}(Q_n + \imath P_n) = F_n^2 + \varepsilon_{n+1}a_{n+1}F_n^3.$$

Let $A \subseteq [n+1]$. If $A \subseteq [n]$ then

$$\widehat{F_{n+1}^1}(A) = \widehat{F_n^2}(A)$$

and by the induction hypothesis (for F_n^2), $|\widehat{F_{n+1}^1}(A)| = \sqrt{2}\prod_{i\in A}|a_i|$. If $A \not\subseteq [n]$ then $n+1 \in A$ and

$$\widehat{F_{n+1}^1}(A) = a_{n+1}\widehat{F_n^3}(A \setminus \{n+1\})$$

and by the induction hypothesis (for F_n^3), $|\widehat{F_{n+1}^1}(A)| = |a_{n+1}|\sqrt{2}\prod_{i\in A\setminus\{n+1\}}|a_i| = \sqrt{2}\prod_{i\in A}|a_i|$. This proves the needed assertion for F_n^1. The other three cases are proved similarly.

It follows from (the proof of) Proposition 3 that

$$H\left(\left|\frac{\widehat{P_n + \imath Q_n}}{\sqrt{2}}\right|^2\right) = -\sum_{i=1}^{n}\left(\prod_{1\le j\le n, j\ne i}(1+a_j^2)\right)a_i^2\log a_i^2.$$

Now, just as in the proof of Theorem 1, using (13), we get the bound

$$H(|\hat{f}_n|^2) > \frac{-1}{1+\max_{1\le i\le n} a_i^2}\sum_{i=1}^{n} a_i^2\log a_i^2.$$

The choice $a_i = 1/\sqrt{n}, i = 1,\ldots,n$, gives

$$I(f_n) < 1 \quad\text{and}\quad H(|\hat{f}_n|^2) > \frac{n}{n+1}\log n.$$

■

Remark 3. By taking a_i-s with higher $\sum a_i^2$ one can get examples of functions of the same kind (L_2 norm 1, L_∞ norm at most $\sqrt{2}$ into $\mathbb{R}$ or into S^1) with influence going to infinity with n and entropy of a larger order of magnitude than the influence. This works as long as the influence is of smaller order of magnitude than its maximal order, n. More precisely, for $1 < a < n$ let $a_i = \sqrt{a}/\sqrt{n}$, $i = 1, \ldots, n$. Then the proofs of Theorems 1 and 2 give functions $f : \{-1,1\}^n \to \mathbb{R}$ and $g : \{-1,1\}^n \to S^1$ with $\|f\|_2 = \|g\|_2 = 1$, $\|f\|_\infty \le \sqrt{2}$, $a/2 < I(f), I(g) < a$ and

$$H(\hat{f}^2), H(|\hat{g}|^2) > \frac{a}{2}(\log n - \log a).$$

So the ratio between the Fourier entropy and the influence tends to infinity with n whenever $a = o(n)$ (and is of order $\log n$ whenever $a < n^c$ for some $c < 1$).

Remark 4. *Joe Neeman pointed out to me that the following is also an example of a well bounded real function with bounded influence and entropy of order* $\log n$*:*

$$f(\varepsilon_1, \ldots, \varepsilon_n) = \begin{cases} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_i\right), & \left|\frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_i\right| \le C \\ C, & \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_i \ge C, \\ -C, & \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_i \le -C. \end{cases}$$

(Stricktly speaking, one needs to normalize f *to have* L_2 *norm exactly 1. This does not cause any complication.)*

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