The number of closed ideals in $L(L_p)$ *

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Abstract

We show that there are $2^{2^{\aleph_0}}$ different closed ideals in the Banach algebra $L(L_p(0,1))$, $1 < p \neq 2 < \infty$. This solves a problem in A. Pietsch's 1978 book "Operator Ideals". The proof is quite different from other methods of producing closed ideals in the space of bounded operators on a Banach space; in particular, the ideals are not contained in the strictly singular operators and yet do not contain projections onto subspaces that are non Hilbertian. We give a criterion for a space with an unconditional basis to have $2^{2^{\aleph_0}}$ closed ideals in terms of the existence of a single operator on the space with some special asymptotic properties. We then show that for $1 < q < 2$ the space $\mathfrak{X}_q$ of Rosenthal, which is isomorphic to a complemented subspace of $L_q(0,1)$, admits such an operator.

1 Introduction

For a reasonably complete discussion of the history of constructing closed ideals in $L(L_p)$, see the introduction in [JPS]. Here we just remark that in 1981, Bourgain, Rosenthal, and the second author [BRS] constructed $\aleph_1$ mutually non isomorphic complemented subspaces of $L_p := L_p(0,1)$ for $1 < p \neq 2 < \infty$, thereby producing (as noted in [P]) $\aleph_1$ different closed ideals in $L(L_p)$. (It is of course well known that the compact operators are the only closed ideal in $L(L_2)$.) At that time it was open whether, absent the continuum hypothesis, $L(L_p)$ contains a continuum of closed ideals. Recently, Schlumprecht and Zsák [SZ] built a continuum of closed ideals in $L_p := L_p(0,1)$.

The main contribution of this paper is Theorem 1 in which we prove that $L(L_p)$, $1 < p \neq 2 < \infty$, has exactly $2^{2^{\aleph_0}}$ different closed ideals.

Recall the notions of small and large closed ideal in $L(X)$. An ideal is called small if it is contained in the ideal of strictly singular operators. Otherwise it is called large. The ideals built in [SZ] are all small, while the ones coming from infinite dimensional

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complemented subspaces are clearly large. Our basic construction is designed to produce large ideals. Note that there are at most a continuum of non mutually isomorphic complemented subspaces of L_p (as the density character of $L(L_p)$ and of the set of projections on L_p is the continuum). So necessarily we produce different kinds of ideals. Unfortunately, we do not produce any new complemented subspaces of L_p.

The new large ideals in $L(L_p)$ that we construct are "smallish" in the sense that, even though there are idempotents in the ideals whose ranges are isomorphic to ℓ_2 (see Remark 5), no operator in any of the ideals is an isomorphism on a copy of ℓ_p. The Kadec–Pełczyński dichotomy principle [KP] implies that every complemented subspace of L_p that is not isomorphic to a Hilbert space contains a complemented subspace that is isomorphic to ℓ_p. Consequently, the range of any infinite rank idempotent in any of the ideals that we construct in Theorem 1 (and, as we said, there are infinite rank idempotents in the ideals) *must* be isomorphic to ℓ_2.

To put these new "smallish" large ideals into perspective within the Banach algebra $L(L_p)$, notice that it follows from the Kadec–Pełczyński dichotomy principle [KP] that there are exactly two different minimal large closed ideals in $L(L_p)$ when $2 < p < \infty$, and thus also for $1 < p < 2$ (because an operator T in $L(L_p)$ is strictly singular if and only if T^* is strictly singular on L_q, $1/p + 1/q = 1$, by Weis' theorem [Wei]). The first of these is $\Gamma_{\ell_p}(L_p)$, the ideal of operators that factor through ℓ_p. This ideal is closed because an operator $T : X \to L_p$, $2 < p < \infty$, factors through ℓ_p if and only if $I_{p,2}T$ is compact, where $I_{p,2}$ is the formal identity mapping from L_p into L_2; see [Joh]. One can prove using the Kadec–Pełczyński dichotomy principle [KP] that $I_{p,2}S$ is compact whenever S is a strictly singular operator on L_p, so the alternate characterization of $\Gamma_{\ell_p}(L_p)$ for $2 < p < \infty$ also yields that $\Gamma_{\ell_p}(L_p)$ contains all strictly singular operators on L_p, $2 < p < \infty$, and thus also for $1 < p < 2$ by [Wei].

The second minimal large closed ideal in $L(L_p)$ is the closure $\overline{\Gamma_2}(L_p)$ of the ideal $\Gamma_2(L_p)$ of operators on L_p that factor through a Hilbert space. Here the closure is needed; in fact, it is not hard to see that there are compact operators on L_p that do not factor through a Hilbert space.

We recall in passing that as was noted in [JPS] the situation in $L(L_1)$ is nicer: $\Gamma_{\ell_1}(L_1)$ is the unique minimal closed large ideal in $L(L_1)$ and it contains all the strictly singular operators on L_1.

In Remark 6 we prove that the new large ideals we construct in $L(L_p)$ do not contain the strictly singular operators on L_p, and hence neither does $\overline{\Gamma_2}(L_p)$. All previously known large ideals in $L(L_p)$ other than $\overline{\Gamma_2}(L_p)$ do contain the strictly singular operators, and this is the first proof that $\overline{\Gamma_2}(L_p)$ does not. A byproduct of Remark 6, stated as Remark 7, is that $L(L_p)$ contains exactly $2^{2^{\aleph_0}}$ small closed ideals.

Our construction and proof of Theorem 1 consist of two steps. In Section 2 we state and prove the technical Proposition 1. This easily yields Corollary 1, which gives a general criterion for a space with an unconditional basis to contain $2^{2^{\aleph_0}}$ different closed ideals. The criterion is in term of the existence of a special operator on the space.

In Section 3 we show that for $1 < q < 2$, the space L_q contains a complemented subspace (this is Rosenthal's $\mathfrak{X}_q$ space, which has an unconditional basis) that admits an operator satisfying the criterion of Proposition 1. The construction here borrows a lot from a previous similar construction from [JS]. Duality and complementation then imply the main result.

2 The main proposition

There is a continuum of infinite subsets of the natural numbers $\mathbb{N}$ each two of which have only finite intersection. Denote some fixed such continuum by $\mathcal{C}$. For a finite dimensional normed space E, we denote by $d(E)$ the Banach–Mazur distance (isomorphism constant) of E to a Euclidean space. Also, recall that, for an operator $T : X \to Y$ between two normed spaces, $\gamma_2(T)$ denotes its factorization constant through a Hilbert space:

$$\gamma_2(T) = \inf\{\|A\|\|B\| \ ; \ T = AB, A : H \to Y, B : X \to H, \ H \text{ a Hilbert space}\}.$$

If T is of rank k, then $\gamma_2(T) \le k^{1/2}\|T\|$ because every k dimensional normed space is $k^{1/2}$-isomorphic to ℓ_2^k [T-J, Theorem 15.5]. Note that $d(E)$ is just $\gamma_2(I_E)$, where I_E is the identity operator on E.

Proposition 1 *Let X be a Banach space with a 1-unconditional basis $\{e_i\}$, let Y be a Banach space, and let $T : X \to Y$ be an operator of norm at most one satisfying:*

(a) For some $\eta > 0$ and for every M there is a finite dimensional subspace E of X such that $d(E) > M$ and $\|Tx\| > \eta\|x\|$ for all $x \in E$.

(b) For some constant Γ and every m there is an n such that every m-dimensional subspace E of $[e_i]_{i\ge n}$ satisfies $\gamma_2(T_{|E}) \le \Gamma$.

Then there exist natural numbers $1 = p_1 < q_1 < p_2 < q_2 < \dots$ such that, denoting for each k, $G_k := [e_i]_{i=p_k}^{q_k}$, and defining for each $\alpha \in \mathcal{C}$, the operator $P_\alpha : X \to [G_k]_{k\in\alpha}$ to be the the natural basis projection, and setting $T_\alpha := TP_\alpha$, we have the following:

If $\alpha_1, \dots, \alpha_s \in \mathcal{C}$ (possibly with repetitions) and $\alpha \in \mathcal{C} \setminus \{\alpha_1, \dots, \alpha_s\}$, then for all $A_1, \dots, A_s \in L(Y)$ and all $B_1, \dots, B_s \in L(X)$,

$$\|T_\alpha - \sum_{i=1}^{s} A_i T_{\alpha_i} B_i\| \ge \eta/2. \tag{1}$$

Proof: Note first that we can strengthen condition (a) to include also that given any n one can chose the subspace E to also satisfy that it is contained in $[e_i]_{i>n}$. Now choose inductively $1 = p_1 < q_1 < p_2 < q_2 \dots$ so that for each k, $G_k = [e_i]_{i=p_k}^{q_k}$ contains a subspace E_k with $\|Tx\| > \eta\|x\|$ for all $x \in E_k$ and

$$d(E_k) \ge q_{k-1}$$

(as we'll see, it is enough that $d(E_k)/q_{k-1}^{1/2} \to \infty$) and, if E is a subspace of $H_k = [G_l]_{l=p_{k+1}}^{\infty}$ with $\dim E \leq q_k$, then

$$\gamma_2(T_{|E}) < \Gamma.$$

Let now $P_\alpha : X \to [G_k]_{k\in\alpha}$ be the natural basis projection and set $T_\alpha := TP_\alpha$. Suppose that $\alpha_1, \ldots, \alpha_s \in \mathcal{C}$ (possibly with repetitions) and $\alpha \in \mathcal{C}\setminus\{\alpha_1, \ldots, \alpha_s\}$. Assume to the contrary that there are $A_1, \ldots, A_s \in L(Y)$ and $B_1, \ldots, B_s \in L(X)$ such that

$$\|T_\alpha - \sum_{i=1}^{s} A_i T_{\alpha_i} B_i\| < \eta/2. \tag{2}$$

There are infinitely many $k \in \alpha \setminus \bigcup_{i=1}^{s} \alpha_i$. For each such k let R_k be the basis projection onto $[G_l]_{l<k}$ and Q_k the basis projection onto $[G_l]_{l>k}$. Now for any $i = 1, \ldots, s$ we have $T_{\alpha_i} G_k = 0$ since $k \notin \alpha_i$, and $\dim(R_k B_i E_k) \leq q_{k-1}$ and $\dim(B_i E_k) \leq q_k$, so we get that for each i,

$$\begin{aligned}\gamma_2(A_i T_{\alpha_i} B_{i|E_k}) &\leq \gamma_2(A_i T_{\alpha_i} R_k B_{i|E_k}) + \gamma_2(A_i T_{\alpha_i} Q_k B_{i|E_k}) \\ &\leq q_{k-1}^{1/2}\|A_i\|\|B_i\| + \Gamma\|A_i\|\|B_i\|.\end{aligned}$$

Consequently,

$$\gamma_2(\sum_{i=1}^{s} A_i T_{\alpha_i} B_{i|E_k}) \leq (\max_{1\leq i\leq s} \|A_i\|\|B_i\|)s(q_{k-1}^{1/2} + \Gamma). \tag{3}$$

On the other hand, since $\|x\| \geq \|T_\alpha x\| \geq \eta\|x\|$ for all $x \in E_k$, (2) implies that

$$(1+\eta/2)\|x\| \geq \|\sum_{i=1}^{s} A_i T_{\alpha_i} B_i x\| \geq \eta\|x\|/2$$

for all $x \in E_k$. Since $d(E_k) \geq q_{k-1}$, we deduce that

$$\gamma_2(\sum_{i=1}^{s} A_i T_{\alpha_i} B_{i|E_k}) \geq \frac{\eta}{2+\eta} q_{k-1}.$$

For k large enough this contradicts (3). ■

Remark 0. Observe that the only condition on T_α that was used to get the inequality (1) is that $\|x\| \geq \|T_\alpha x\| \geq \eta\|x\|$ for all x in E_k with $k \in \alpha$. Consequently, the proof of Corollary 1 below shows that any operator S in $L(X)$ for which there is $\eta > 0$ such that $\|Sx\| \geq \eta\|x\|$ for all x in E_k with $k \in \alpha$ cannot be in the closed ideal generated by $\{T_\beta : \beta \in \mathcal{C}, \beta \neq \alpha\}$. In fact, from the proof of Proposition 1, only the inequality $\|Sx\| \geq \eta\|x\|$ for all x in H_k with $k \in \alpha$ and where H_k is isomorphic to E_k with isomorphism constant independent of k is sufficient to conclude that S is not in the closed ideal generated by $\{T_\beta : \beta \in \mathcal{C}, \beta \neq \alpha\}$. This observation is used in Remark 6 at the end of this paper.

Corollary 1 *Let X be a Banach space with a 1-unconditional basis $\{e_i\}$ and assume there is an operator $T : X \to X$ of norm at most one satisfying (a) and (b) of Proposition 1. Then $L(X)$ has exactly $2^{2^{\aleph_0}}$ different closed ideals.*

Proof: For any nonempty proper subset $\mathcal{A}$ of $\mathcal{C}$ let $\mathcal{I}_{\mathcal{A}}$ be the ideal generated by $\{T_\alpha\}_{\alpha\in\mathcal{A}}$; i.e., all operators of the form $\sum_{i=1}^{s} A_i T_{\alpha_i} B_i$ with $s \in \mathbb{N}$, $A_i, B_i \in L(X)$, $\alpha_i \in \mathcal{A}$, $i = 1,\dots,s$. To avoid cumbersome notation, interpret $\mathcal{A} \subset \mathcal{C}$ to mean that $\mathcal{A}$ is a nonempty proper subset of $\mathcal{C}$.

Since we allow repetition of the T_{α_i}, it is easy to see that this really defines a (non closed) ideal. Let $\mathcal{B}$ be a subset of $\mathcal{C}$ different from $\mathcal{A}$ and assume, without loss of generality, that $\mathcal{B} \not\subset \mathcal{A}$. Let $\alpha \in \mathcal{B}\setminus\mathcal{A}$. Then by Proposition 1, $T_\alpha \notin \overline{\mathcal{I}_{\mathcal{A}}}$. Consequently, $\{\overline{\mathcal{I}_{\mathcal{A}}}\}_{\mathcal{A}\subset\mathcal{C}}$ are all different.

Since the density character of $L(X)$, for any separable X, is at most the continuum, it is easy to see that, for any separable space X, $L(X)$ has at most $2^{2^{\aleph_0}}$ different closed ideals. ∎

Remarks:

1. One can strengthen the conclusion of the corollary by getting an antichain of $2^{2^{\aleph_0}}$ closed ideals in $L(X)$; i.e., such a collection no two of whose members are included one in the other. For that one just uses a collection of $2^{2^{\aleph_0}}$ subsets of $\mathcal{C}$ no two of which are included one in the other.

2. Similarly, one gets a collection of $2^{\aleph_0}$ different closed ideals in $L(X)$ that form a chain (by taking a chain of subsets of $\mathcal{C}$ of that cardinality). It is also easy to show by a density argument that, for any separable X, this is the maximal cardinality of any chain of closed ideals in $L(X)$.

3. If Y is a Banach space that contains a complemented subspace X with the properties of Corollary 1 then clearly $L(Y)$ also has $2^{2^{\aleph_0}}$ different closed ideals (actually an antichain). The same is true also for any space isomorphic to such a Y.

4. The simplest examples of spaces X that satisfy the hypotheses of Corollary 1 and thus $L(X)$ has $2^{2^{\aleph_0}}$ different closed ideals are $(\sum \ell_{r_i}^{n_i})_2$ for $r_i \uparrow 2$ and n_i satisfying $n_i^{\frac{1}{r_i}-\frac{1}{2}} \to \infty$. Consequently, by Remark 3, $L((\sum \ell_{r_i})_2)$ for $r_i \uparrow 2$ also has $2^{2^{\aleph_0}}$ different closed ideals. Interesting, but less natural, examples of separable spaces X with $L(X)$ having $2^{2^{\aleph_0}}$ different closed ideals were known before (see [M]). Unfortunately $(\sum \ell_{r_i}^{n_i})_2$ for $r_i \uparrow 2$ and $n_i^{\frac{1}{r_i}-\frac{1}{2}} \to \infty$ does not embed isomorphically as a complemented subspace into any L_p, $p < \infty$, so this example is not good for our purposes. Actually, at least for some sequences $\{(r_i, n_i)\}$ with the above properties, $(\sum \ell_{r_i}^{n_i})_2$ does not even embed isomorphically into any L_p space, $p < \infty$. That this is true, for example, if each $(r, n) \in \{(r_i, n_i)\}$ repeats n times follows from Corollary 3.4 in [KS].

In the next section we show how to get complemented subspaces of the reflexive L_p spaces that satisfy the hypotheses of Corollary 1.

3 The Operator T

In this section we prove that for each $1 < q < 2$ there is a complemented subspace of L_q isomorphic to a space X with a 1-unconditional basis on which there is an operator of norm at most one with properties (a) and (b) of Proposition 1.

Recall that for a sequence $u = \{u_j\}_{j=1}^{\infty}$ of positive real numbers and for $p > 2$, the Banach space $\mathfrak{X}_{p,u}$ is the sequence space with norm

$$\|\{a_j\}_{j=1}^{\infty}\| = \max\{(\sum_{j=1}^{\infty} |a_j|^p)^{1/p}, (\sum_{j=1}^{\infty} |a_j u_j|^2)^{1/2}\}. \tag{4}$$

Rosenthal [Ro1] proved that $\mathfrak{X}_{p,u}$ is isomorphic to a complemented subspace of L_p with the isomorphism constant and the complementation constant depending only on p. If u is such that for all $\varepsilon > 0$, $\sum_{u_j<\varepsilon} u_j^{\frac{2p}{p-2}} = \infty$, then one gets a space isomorphically different from ℓ_p, ℓ_2 and $\ell_p \oplus \ell_2$. However, for different u satisfying the condition above, the different $\mathfrak{X}_{p,u}$ spaces are mutually isomorphic. We denote by $\mathfrak{X}_p$ any of these spaces. Later we shall need more properties of the spaces $\mathfrak{X}_{p,u}$ and of particular embeddings of them into L_p, but for now we only need the representation (4) and we think of $\mathfrak{X}_{p,u}$ as a subspace of $\ell_p \oplus_\infty \ell_2$.

Let $\{e_j\}_{j=1}^{\infty}$ be the unit vector basis of ℓ_p and let $\{f_j\}_{j=1}^{\infty}$ be the unit vector basis of ℓ_2. Let $v = \{v_j\}_{j=1}^{\infty}$ and $w = \{w_j\}_{j=1}^{\infty}$ be two positive real sequences such that $\delta_j = w_j/v_j \to 0$ as $j \to \infty$ and $\max_{1\le j<\infty} \delta_j \le 1$. Set

$$g_j^v = e_j + v_j f_j \in \ell_p \oplus_\infty \ell_2 \quad \text{and} \quad g_j^w = e_j + w_j f_j \in \ell_p \oplus_\infty \ell_2.$$

Then $\{g_j^v\}_{j=1}^{\infty}$ is the unit vector basis of $\mathfrak{X}_{p,v}$ and $\{g_j^w\}_{j=1}^{\infty}$ is the unit vector basis of $\mathfrak{X}_{p,w}$. Define also

$$\Delta : \mathfrak{X}_{p,w} \to \mathfrak{X}_{p,v}$$

by

$$\Delta g_j^w = \delta_j g_j^v.$$

Note that Δ is the restriction to $\mathfrak{X}_{p,w}$ of $K \in L(\ell_p \oplus_\infty \ell_2)$ defined by

$$K(e_j) = \delta_j e_j \quad \text{and} \quad K(f_j) = f_j$$

Consequently, $\|\Delta\| \le \|K\| = 1$.

The following proposition follows immediately from the easily verified fact that $\|K_{|[e_j]_{j=m}^{\infty}}\| \to 0$ as $m \to \infty$.

Proposition 2 *Given n there exists an m such that if E is an n dimensional subspace of $[e_j]_{j=m}^{\infty} \oplus [f_j]_{j=1}^{\infty} \subset \ell_p \oplus_\infty \ell_2$, then $\gamma_2(K_{|E}) \le 2$. In particular, if E is an n dimensional subspace of $[g_j^w]_{j=m}^{\infty} \subset \mathfrak{X}_{p,w}$, then $\gamma_2(\Delta_{|E}) \le 2$.*

Next we define weights $\{v_j\}$ and $\{w_j\}$ with some additional properties. For that we use different representations of the spaces $\mathfrak{X}_{p,u}$. It was proved in [Ro1] that if $\{X_j\}_{j=1}^{\infty}$, is a sequence of symmetric, each three valued, independent random variables all L_p normalized, $2 < p < \infty$, then $\{X_j\}_{j=1}^{\infty}$ is equivalent, in L_p, to $\{g_j^u\}_{j=1}^{\infty}$, the unit vector basis of $\mathfrak{X}_{p,u}$, where $u_j = \|X_j\|_2$. Defining $Y_j = X_j/\|X_j\|_q$, for $q = p/(p-1)$, $\{Y_j\}_{j=1}^{\infty}$ is equivalent, in L_q, to the basis $\{h_j^u\}_{j=1}^{\infty}$ of $\mathfrak{X}_{q,u} := \mathfrak{X}_{p,u}^*$ that is dual to the unit vector basis of $\mathfrak{X}_{p,u}$.

Let us say already at this early stage that, for some appropriate weights $\{v_j\}$ and $\{w_j\}$, the operator T we are after will be of the form Δ^* followed by a norm one isomorphism from $\mathfrak{X}_{q,w}$ to $\mathfrak{X}_{q,v}$.

Recall that $P : L_p \to [X_j]_{j=1}^{\infty}$ defined by

$$Pf = \sum_{j=1}^{\infty}(\int_0^1 fY_j)X_j$$

defines a bounded projection onto $[X_j]_{j=1}^{\infty}$ (and P^* a bounded projection from L_q onto $[Y_j]_{j=1}^{\infty}$). The norms of the equivalences above and of the projections depend on p but not on the particular weights u.

We now recall a construction from Section 4 of [JS]. It was shown there that, given $1 < q < 2$, any sequence $\{\delta_i\}_{i=1}^{\infty}$ that decreases to zero, any sequence $\{r_i\}_{i=1}^{\infty}$ such that $q < r_i \uparrow 2$ fast enough and in particular satisfying $\delta_i^{\frac{q(2-r_i)}{2-q}} > 1/2$, $i = 1, 2, \dots$, and for any sequence $\varepsilon_i \downarrow 0$, we can find two sequences $\{Y_i\}$ and $\{Z_i\}$ of symmetric, independent, three valued random variables, all normalized in L_q, with the following additional properties:

- Put $v_j = 1/\|Y_j\|_2$ and $w_j = 1/\|Z_j\|_2$. Then there are disjoint finite subsets σ_i, $i = 1, 2, \dots$, of the integers such that $w_j = \delta_i v_j$ for $j \in \sigma_i$.
- There are independent random variables $\{\bar{Y}_i\}$ and $\{\bar{Z}_i\}$, with $\bar{Y}_i$ normalized in L_q and r_i–stable; $\bar{Z}_i$ is r_i–stable with $1 \geq \|\bar{Z}_i\|_q \geq 3/4$ for each i, and there are coefficients $\{a_j\}$ such that

$$\|\bar{Y}_i - \sum_{j\in\sigma_i} a_j Y_j\|_q < \varepsilon_i \quad \text{and} \quad \|\bar{Z}_i - \sum_{j\in\sigma_i} \delta_i a_j Z_j\|_q < \varepsilon_i. \tag{5}$$

We may of course repeat each of the triplets $r_i, \delta_i, \varepsilon_i$-s as many (finitely many) times as we wish. Thus we conclude that given any sequence $\{\delta_i\}_{i=1}^{\infty}$ decreasing to zero, any sequence $\{r_i\}_{i=1}^{\infty}$ such that $q < r_i \uparrow 2$ and satisfying $\delta_i^{\frac{q(2-r_i)}{2-q}} > 1/2$, $i = 1, 2, \dots$, any sequence of integers n_i, and any sequence $\varepsilon_i \downarrow 0$, we can find two sequences $\{Y_i\}$ and $\{Z_i\}$ of symmetric, independent, three valued random variables, all normalized in L_q, with the following additional properties:

- Put $v_j = 1/\|Y_j\|_2$ and $w_j = 1/\|Z_j\|_2$. Then there are disjoint finite subsets $\sigma_{i,l}$, $i = 1, 2, \dots, l = 1, \dots n_i$ of the integers such that $w_j = \delta_i v_j$ for $j \in \sigma_{i,l}$.

- There are independent random variables $\{\bar{Y}_{i,l}\}$ r_i–stable normalized in L_q, $\{\bar{Z}_{i,l}\}$ r_i–stable with $1 \geq \|\bar{Z}_{i,l}\|_q \geq 3/4$ for each i and l, and there are coefficients $\{a_j\}$ such that

$$\|\bar{Y}_{i,l} - \sum_{j\in\sigma_{i,l}} a_j Y_j\|_q < \varepsilon_i \quad \text{and} \quad \|\bar{Z}_{i,l} - \sum_{j\in\sigma_{i,l}} \delta_i a_j Z_j\|_q < \varepsilon_i. \tag{6}$$

Choosing the ε_i small enough, we can assume that $\{\sum_{j\in\sigma_{i,l}} a_j Y_j\}_{l=1}^{n_i}$ is, in L_q, 2-equivalent to the unit vector basis of $\ell_{r_i}^{n_i}$, and similarly $\{\sum_{j\in\sigma_{i,l}} \delta_i a_j Z_j\}_{l=1}^{n_i}$ is, in L_q, 2-equivalent to the unit vector basis of $\ell_{r_i}^{n_i}$. Denoting by R the map that sends Y_j to $\delta_i Z_j$ for $j \in \sigma_{i,l}$, we get that this map satisfies that for all i there is a space E_i that is 2-isomorphic to $\ell_{r_i}^{n_i}$ such that $\|Rx\| \geq \|x\|/4$ for all $x \in E_i$. Choosing the n_i large enough, we can also assume that for all k,

$$n_i^{\frac{1}{r_i}-\frac{1}{2}} \to \infty \quad \text{as} \quad i \to \infty.$$

Since $n_i^{\frac{1}{r_i}-\frac{1}{2}}$ is the distance of $\ell_{r_i}^{n_i}$ to a Hilbert space, we get that $d(E_i) \to \infty$.

We are now ready to state and prove the main proposition of this section.

Proposition 3 *With the choice of $v = \{v_j\}$ and $w = \{w_j\}$ above, set $X = \mathfrak{X}_{q,v}$, let $\Delta^* : \mathfrak{X}_{q,v} \to \mathfrak{X}_{q,w}$ be the adjoint of Δ defined at the beginning of this section, and let S be a norm one isomorphism from $\mathfrak{X}_{q,w}$ onto $\mathfrak{X}_{q,v}$. Put $T = S\Delta^*$. Then X, T satisfy the assumptions of Proposition 1.*

Proof: Since $T = ARB$ for isomorphisms A and B, the discussion above provides a proof of property (a). Property (b) follows by duality from Proposition 2. Indeed, fix m and n and let E be an m-dimensional subspace of $[h_i^v]_{i\geq n}$. $\Delta^*(E)$ is a subspace of $[h_i^w]_{i\geq n}$, so there is a $k = k(m)$-dimensional subspace F of $[g_i^w]_{i\geq n}$ that 2-norms E. Here, $k = k(m)$ depends only on m (and we used the 1-unconditionality of the bases). By Proposition 2, for some n depending only on k and thus only on m, $\gamma_2(\Delta_{|F}) \leq 2$. From this it is easy to get that $\gamma_2(\Delta^*_{|E}) \leq 4$. Consequently, this holds also for $T = S\Delta^*$. ■

4 The main result and additional comments

Theorem 1 *For every $1 < p \neq 2 < \infty$ the number of different closed ideals in $L(\mathfrak{X}_p)$ and in $L(L_p)$ is exactly $2^{2^{\aleph_0}}$. Moreover, each of these spaces contains an antichain of closed ideals of cardinality $2^{2^{\aleph_0}}$ and a chain of cardinality $2^{\aleph_0}$.*

Proof: For $\mathfrak{X}_q$, $1 < q < 2$, the theorem follows from Proposition 3 and Corollary 1. For $\mathfrak{X}_p$, $2 < p < \infty$, it follows by simple duality. Since for $1 < p \neq 2 < \infty$ the space $\mathfrak{X}_p$ is isomorphic to a complemented subspace of L_p, it follows also for L_p.

The statements about chains and antichains follow from the remarks at the end of Section 2. ■

Remark 5. As is stated in the introduction, the new ideals in $L(L_p)$ and $L(\mathfrak{X}_p)$, $1 < p \neq 2 < \infty$, constructed in Theorem 1 are all large and in fact contain projections whose ranges are isomorphic to ℓ_2.

Proof: First we observe that it is enough to show that for each $\alpha \in \mathcal{C}$, the operator T_α on X (recall that X is isomorphic to $\mathfrak{X}_q$, where $1 < q < 2$), isomorphically preserves a copy of ℓ_2. Here T is the operator produced in Proposition 3 and T_α is defined in the statement of Proposition 1. Indeed, since any subspace of L_q, $1 < q < 2$, that is isomorphic to ℓ_2 contains a further infinite dimensional subspace that is complemented in L_q (this fact was probably first observed by Pełczyński; see [JS, p. 1106] for a proof), this will show that the identity on ℓ_2 factors through T_α and hence there is a projection in the ideal generated by T_α whose range is isomorphic to ℓ_2. This will give Remark 5 for $L(\mathfrak{X}_p)$ when $1 < p < 2$ and the case of $L(\mathfrak{X}_p)$ for $2 < p < \infty$ follows by duality. The statement for $L(L_p)$, $1 < p \neq 2 < \infty$, is then immediate.

To show that T_α isomorphically preserves a copy of ℓ_2, note that the space $\mathfrak{X}_{q,v}$ we built contains a modular space [LT, Def. 4.d.1] $\ell_{\{r_i\}}$ with $r_i \uparrow 2$ on which T_α is an isomorphism and thus (by passing to a subsequence of the sequence r_i that tends quickly to 2), also contains an isomorph of ℓ_2 on which T_α is an isomorphism. ■

Remark 6. The large ideals in $L(L_q)$ and $L(\mathfrak{X}_q)$ constructed in Theorem 1 do not contain the ideal of strictly singular operators.

Proof: (sketch): By [Wei] and how we constructed the ideals in $L(L_q)$ from the ideals in $L(\mathfrak{X}_q)$, it is enough to consider the ideals constructed in $L(\mathfrak{X}_q)$ for $1 < q < 2$. Let T be the operator and X be the space isomorphic to $\mathfrak{X}_q$ that are defined in Proposition 3 and which satisfy the assumptions of Proposition 1. Let $\{T_\alpha : \alpha \in \mathcal{C}\}$ be the corresponding operators on X given by Proposition 3. As in the proof of Corollary 1, for $\mathcal{A}$ a (always nonempty, proper) subset of $\mathcal{C}$ let $\mathcal{I}_\mathcal{A}$ be the ideal in $L(X)$ generated by $\mathcal{A}$. Given $\mathcal{A} \subset \mathcal{C}$, take any $\alpha \in \mathcal{C}$ that is not in $\mathcal{A}$. We know that T_α is not in $\overline{\mathcal{I}}_\mathcal{A}$, but we want a strictly singular operator that is not in $\overline{\mathcal{I}}_\mathcal{A}$ and T_α is not strictly singular. Let $Y := (\sum_{k=1}^\infty G_k)_q$, where the G_k are the block subspaces of X defined in the proof of Proposition 1. The G_k are contractively complemented in X and X is isomorphic to a complemented subspace of L_q, hence Y is isomorphic to a complemented subspace of ℓ_q (and thus to ℓ_q by Pełczyński's well-known theorem, but we do not need this) which in turn is isomorphic to a complemented subspace of X. Define $U : Y \to X$ by making U the identity on each G_k and extending by linearity and continuity. This is OK because L_q has type q and (G_k) is a monotonely unconditional Schauder decomposition for a subspace of X, hence the decomposition (G_k) has an upper q-estimate (even with constant 1). Let E_k be the subspace of G_k defined in the proof of Proposition 1. The operator $T_\alpha U$ is strictly singular and $\|T_\alpha U x\| \geq \eta \|x\|$ for all x in E_k with k in α. Since Y is isomorphic to a complemented subspace of X, we also get a strictly singular operator $S : X \to X$ and subspaces H_k of Y with H_k isomorphic to E_k (with isomorphism constant independent of

k) such $\|Sx\| \geq \eta\|x\|$ for all $x \in H_k$ with $k \in \alpha$. By Remark 0 after Proposition 1, this is enough to yield that S is not in the closed ideal in $L(X)$ generated by $\{T_\beta : \beta \in \mathcal{C}, \beta \neq \alpha\}$. ■

Remark 7. $L(L_q)$ and $L(\mathfrak{X}_q)$, $1 < q \neq 2 < \infty$ both contain exactly $2^{2^{\aleph_0}}$ closed small ideals.

Proof: (sketch): Again, it is enough to deal with the case of $L(\mathfrak{X}_q)$ with $1 < q < 2$. Let X and T be as in Remark 6. For $\mathcal{A} \subset \mathcal{C}$, let $\mathcal{J}_\mathcal{A}$ be the ideal in $L(X)$ generated by $\{T_\alpha UP : \alpha \in \mathcal{A}\}$, where P is any fixed projection from X onto a subspace isomorphic to Y (we identify Y with that subspace). All $\overline{\mathcal{J}_\mathcal{A}}$ are small ideals and clearly $\mathcal{J}_\mathcal{A}$ is contained in the ideal $I_\mathcal{A}$ generated by $\{T_\alpha : \alpha \in \mathcal{A}\}$. But in Remark 6 we saw that $T_\alpha UP$ is not contained in $\overline{I}_\mathcal{A}$ when $\alpha \notin \mathcal{A}$, so $\overline{\mathcal{J}}_\mathcal{A} \neq \overline{\mathcal{J}}_\mathcal{B}$ when $A \neq B$. ■

References

[BRS] Bourgain, J.; Rosenthal, H. P.; Schechtman, G. An ordinal L_p-index for Banach spaces, with application to complemented subspaces of L_p. Ann. of Math. (2) 114 (1981), no. 2, 193–228.

[Joh] Johnson, William B. Operators into L_p which factor through ℓ_p. J. London Math. Soc. (2) 14 (1976), no. 2, 333–339.

[JPS] Johnson, William B.; Pisier, Gilles; Schechtman, Gideon, Ideals in $L(L_1)$, Math. Ann. 376 (2020), no. 1–2, 693—705.

[JS] Johnson, William B.; Schechtman, Gideon, Multiplication operators on $L(L_p)$ and ℓ_p-strictly singular operators. J. Eur. Math. Soc. (JEMS) 10 (2008), no. 4, 1105—1119.

[KP] Kadec, M. I.; Pełczyński, A. Bases, lacunary sequences and complemented subspaces in the spaces L_p. Studia Math. 21 1961/1962 161–176.

[KS] Kwapień, Stanisław; Schütt, Carsten Some combinatorial and probabilistic inequalities and their application to Banach space theory. II. Studia Math. 95 (1989), no. 2, 141–154.

[LT] Lindenstrauss, J. and Tzafriri, L. Classical Banach spaces I: Sequence spaces. Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 92. Springer-Verlag, Berlin-New York, (1977).

[M] Mankiewicz, P. A superreflexive Banach space X with $L(X)$ admitting a homomorphism onto the Banach algebra $C(\beta\mathbb{N})$. Israel J. Math. 65 (1989), no. 1, 1–16.

[P] Pietsch, A. Operator Ideals. Volume 16 of Mathematische Monographien. Deutscher Verlag der Wissenschaften (1978).

[Ro1] Rosenthal, Haskell P. On the subspaces of L^p $(p > 2)$ spanned by sequences of independent random variables. Israel J. Math. 8 (1970), 273–303.

[SZ] Schlumprecht, Thomas; Zsák, András. The algebra of bounded linear operators on $\ell_p \oplus \ell_q$ has infinitely many closed ideals. J. Reine Angew. Math. 735 (2018), 225–247.

[T-J] N. Tomczak-Jaegermann, *Banach-Mazur distances and finite-dimensional operator ideals,* Pitman Monographs and Surveys in Pure and Applied Mathematics **38**, Longman (1989).

[Wei] Weis, L. On perturbations of Fredholm operators in $L_p(\mu)$-spaces. Proc. Amer. Math. Soc. 67 (1977), no. 2, 287–292.

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