Eidelheit's Theorem Revisited

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Based on an observation by
Johnson, Phillips and Schechtman

Eidelheit's Theorem

We say that two Banach algebras $\mathcal{A}$ and $\mathcal{B}$ are isomorphic as Banach algebras if there is an injective and surjective homomorphism $\Phi : \mathcal{A} \to \mathcal{B}$ such that $\|\Phi\|, \|\Phi^{-1}\| < \infty$. Φ is called a Banach algebras isomorphism.

For a Banach space X, $L(X)$ denotes the Banach algebra of all bounded linear operators on X.

A theorem of Meier Eidelheit from 1940 states that if $L(X)$ and $L(Y)$ are isomorphic as Banach algebras then X and Y are isomorphic as Banach spaces. Moreover, if $\Phi : L(X) \to L(Y)$ is the Banach algebra isomorphism then there is a Banach space isomorphism $U : X \twoheadrightarrow Y$ such that for all $A \in L(X)$

$$\Phi(A) = UAU^{-1}.$$

Eidelheit

Meier Eidelheit, 1910–1943, was a student of Banach in Lwów (=Lvov=Lviv).
He was murder in the Holocaust.

We (Bill Johnson, Chris Phillips and I) noticed that digging a bit into the proof of Eidelheit theorem one can prove a stronger theorem.

Main Theorem

Theorem. (JPS)

Let $\mathcal{A}$ be a Banach subalgebra of $L(X)$ and $\mathcal{B}$ a Banach subalgebra of $L(Y)$. Assume that $\mathcal{A} \supseteq F(X)$ (the finite rank operators) and that $\mathcal{B} \supseteq F(Y)$. Assume that

$$\Phi : \mathcal{A} \twoheadrightarrow \mathcal{B}$$

is a Banach algebra isomorphism.Then, there is an isomorphism $U : X \twoheadrightarrow Y$ such that for all $A \in \mathcal{A}$

$$\Phi(A) = UAU^{-1}.$$

Moreover $\|U\| \le \|\Phi\|$ and $\|U^{-1}\| \le \|\Phi^{-1}\|$.

Main Theorem 2

Actually, as we'll see, the continuity of Φ and Φ^{-1} is automatic and need not be assumed so a stronger theorem holds

Theorem. (JPS)

Let $\mathcal{A}$ be a Banach subalgebra of $L(X)$ and $\mathcal{B}$ a Banach subalgebra of $L(Y)$. Assume that $\mathcal{A} \supseteq F(X)$ (the finite rank operators) and that $\mathcal{B} \supseteq F(Y)$. Assume that

$$\Phi : \mathcal{A} \twoheadrightarrow \mathcal{B}$$

is an homomorphism of algebras .Then, Φ and Φ^{-1} are bounded and there is an isomorphism $U : X \twoheadrightarrow Y$ such that for all $A \in \mathcal{A}$

$$\Phi(A) = UAU^{-1}.$$

Moreover $\|U\| \leq \|\Phi\|$ and $\|U^{-1}\| \leq \|\Phi^{-1}\|$.

Corollary

Recall that a (two sided) ideal in $L(X)$ is a subspace $\mathcal{I}$ such that for all $A \in \mathcal{I}$ and $B \in L(X)$ AB and BA are in $\mathcal{I}$.

Corollary.

If $\mathcal{I}$ and $\mathcal{J}$ are closed ideals in $L(X)$ which are isomorphic as Banach algebras then $\mathcal{I} = \mathcal{J}$.

Proof: Assume $\mathcal{I}$ and $\mathcal{J}$ are isomorphic as Banach algebras. Since any ideal in $L(X)$ contains $F(X)$ it follows that there is a Banach space isomorphism U of X onto itself such that $A \to UAU^{-1}$ maps $\mathcal{I}$ onto $\mathcal{J}$.
If $B \in \mathcal{J}$ there is an $A \in \mathcal{I}$ such that

$$B = UAU^{-1}.$$

Since $\mathcal{I}$ is an ideal $B \in \mathcal{I}$. So $\mathcal{J} \subseteq \mathcal{I}$. Similarly, $\mathcal{I} \subseteq \mathcal{J}$. ■

Remark: Enough to assume that $\mathcal{I}$ and $\mathcal{J}$ are homomorphic as algebras.

boosting

This corollary can be used to boost several recent theorems concerning the number of closed ideals in $L(X)$ for some classical spaces X.

Theorem. (JPS +)

There is a continuum of closed ideals in $L(L_1(0,1))$ each two of which are non-isomorphic as Banach algebras. The same holds for $L(C(0,1))$ and $L(L_\infty(0,1))$.

Theorem. (JS +)

For each $1 < p \neq 2 < \infty$ there are exactly $2^{continuum}$ closed ideals in $L(L_p(0,1))$ up to isomorphism of Banach algebras.

Theorem. (Freeman,Schlumprecht,Zsak +)

For $1 < p < q < \infty$ there are exactly $2^{continuum}$ closed ideals in $L(\ell_p \oplus \ell_q)$ up to isomorphism of Banach algebras. Same holds for $L(\ell_1 \oplus \ell_q)$ and $L(\ell_p \oplus c_0)$.

Remark: In the last three theorems one can replace “isomorphism of Banach algebras" by “homomorphism of algebras".

In the rest of this talk I'll give the, hopefully complete, proof of the Theorem.

Lemma

Lemma

The following are equivalent for $V_0 \in L(X)$:
1. V_0 is of rank at most one.
2. For all $V \in L(X)$ there is a $\lambda \in \mathbb{C}$ such that

$$(VV_0)^2 = \lambda VV_0$$

3. For all $V \in L(X)$ of rank 2 there is a $\lambda \in \mathbb{C}$ such that

$$(VV_0)^2 = \lambda VV_0.$$

Proof. $1 \Rightarrow 2$:
Let V_0 be of rank 1. For each $V \in L(X)$, $W = VV_0$ is of rank at most 1. Assume it is of rank 1. $W = x^* \otimes x$, $W(y) = x^*(y)x$.

$$W^2(y) = x^*(y)W(x) = x^*(y)x^*(x)x = x^*(x)W(y).$$

Lemma

$2 \Rightarrow 3$ is clear.

$3 \Rightarrow 1$:

Assume that the range of V_0 contains two independent vectors y_1 and y_2. Let x_1 and x_2 be such that

$$V_0(x_1) = y_1, \qquad V_0(x_2) = y_2.$$

Let f_1 and f_2 be linear functionals such that

$$f_1(y_1) = 1, \ \ f_1(y_2) = 0, \ \ f_2(y_1) = 0, \ \ f_2(y_2) = 1$$

and put

$$V(x) = f_1(x)x_2 + f_2(x)x_1.$$

Then,

$$VV_0(x_1) = x_2, \quad VV_0(x_2) = x_1.$$

Lemma

$$VV_0(x_1) = x_2, \quad VV_0(x_2) = x_1.$$

Since, $(VV_0)^2 = \lambda VV_0$,

$$x_1 = (VV_0)^2(x_1) = \lambda VV_0(x_1) = \lambda x_2$$

and thus

$$y_1 = \lambda y_2.$$

A contradiction. ■

Proof of the Theorem

Theorem. (JPS)

Let $\mathcal{A}$ be a Banach subalgebra of $L(X)$ and $\mathcal{B}$ a Banach subalgebra of $L(Y)$. Assume that $\mathcal{A} \supseteq F(X)$ (the finite rank operators) and that $\mathcal{B} \supseteq F(Y)$. Assume that

$$\Phi : \mathcal{A} \twoheadrightarrow \mathcal{B}$$

is a Banach algebra isomorphism.Then, there is an isomorphism $U : X \twoheadrightarrow Y$ such that for all $A \in \mathcal{A}$

$$\Phi(A) = UAU^{-1}.$$

Moreover $\|U\| \leq \|\Phi\|$ and $\|U^{-1}\| \leq \|\Phi^{-1}\|$.

Proof of the Theorem

Let V_0 be a rank one projection:

$$V_0(x) = f_0(x)x_0$$

where $x_0 \in X,\ \ f_0 \in X^*,\ \ \|x_0\| = \|f_0\| = 1,\ \ \ f_0(x_0) = 1$.
By the Lemma for all $V \in \mathcal{A}$ there is a $\lambda \in \mathbb{C}$ such that

$$(VV_0)^2 = \lambda VV_0.$$

Applying Φ we get that for all $W \in \mathcal{B}$ there is a $\lambda \in \mathbb{C}$ such that

$$(W\Phi(V_0))^2 = \lambda W\Phi(V_0).$$

In particular this holds for all $W \in L(Y)$ of rank 2. So, By the Lemma $W_0 = \Phi(V_0)$ is of rank one. Say,

$$W_0(y) = g_0(y)y_0,$$

for some $y_0 \in Y,\ \ g_0 \in Y^*,\ \ \|y_0\| = 1,\ \ \|g_0\| \leq \|\Phi\|$.

Proof of the Theorem

$$V_0(x) = f_0(x)x_0, \quad W_0(y) = \Phi(V_0)(y) = g_0(y)y_0.$$

We now define the isomorphism $U : X \twoheadrightarrow Y$.
Given a $z \in X$ choose an *arbitrary* $V \in \mathcal{A}$ such that $V(x_0) = z$ and define

$$U(z) = \Phi(V)(y_0).$$

We need to show:
a. U is well defined (doesn't depend on the choice of V)
b. U is injective
c. U is surjective
d. U is linear
e. U is bounded, $\|U\| \leq \|\Phi\|$
f. U^{-1} is bounded, $\|U^{-1}\| \leq \|\Phi^{-1}\|$
g. For all $A \in \mathcal{A}$,

$$\Phi(A) = UAU^{-1}.$$

$$V_0(x) = f_0(x)x_0, \quad W_0(y) = \Phi(V_0)(y) = g_0(y)y_0.$$

$$U(z) = \Phi(V)(y_0) \text{ for } V \text{ such that } V(x_0) = z.$$

Assume $V_1(x_0) = V_2(x_0) = z, \quad V_1, V_2 \in \mathcal{A}$. Then, for all $x \in X$,

$$V_1(V_0(x)) = f_0(x)z = V_2(V_0(x)).$$

So, $V_1 V_0 = V_2 V_0$
and $\Phi(V_1)W_0 = \Phi(V_2)W_0$, i.e., for all $y \in Y$,

$$g_0(y)\Phi(V_1)(y_0) = g_0(y)\Phi(V_2)(y_0).$$

So,

$$\Phi(V_1)(y_0) = \Phi(V_2)(y_0). \quad \blacksquare$$

Proof of the Theorem, b. Injectivity

$$V_0(x) = f_0(x)x_0, \quad W_0(y) = \Phi(V_0)(y) = g_0(y)y_0.$$

$$U(z) = \Phi(V)(y_0) \text{ for } V \text{ such that } V(x_0) = z.$$

Let $z_1 \neq z_2$. Let $V_1, V_2 \in \mathcal{A}$ such that

$$V_1(x_0) = z_1, \quad V_2(x_0) = z_2.$$

Then $V_1 V_0 \neq V_2 V_0$, so,

$$\Phi(V_1)W_0 \neq \Phi(V_2)W_0$$

Implying

$$\Phi(V_1)(y_0) \neq \Phi(V_2)(y_0)$$

i.e.,

$$U(z_1) \neq U(z_2). \quad \blacksquare$$

$$V_0(x) = f_0(x)x_0, \quad W_0(y) = \Phi(V_0)(y) = g_0(y)y_0.$$

$$U(z) = \Phi(V)(y_0) \text{ for } V \text{ such that } V(x_0) = z.$$

Let $y \in Y$ and let W be a rank one operator in $\mathcal{B}$ such that $W(y_0) = y$. Let $V \in \mathcal{A}$ be such that $\Phi(V) = W$ and let $z = V(x_0)$. Then,

$$U(z) = \Phi(V)(y_0) = W(y_0) = y. \quad \blacksquare$$

$$V_0(x) = f_0(x)x_0, \quad W_0(y) = \Phi(V_0)(y) = g_0(y)y_0.$$

$U(z) = \Phi(V)(y_0)$ for V such that $V(x_0) = z$.

Let $z_1, z_2 \in X$, $\lambda \in \mathbb{C}$. Let $V_1(x_0) = z_1$, $V_2(x_0) = z_2$. Then, $(V_1 + V_2)(x_0) = z_1 + z_2$, so,

$$U(z_1+z_2) = \Phi(V_1+V_2)(y_0) = \Phi(V_1)(y_0)+\Phi(V_2)(y_0) = U(z_1)+U(z_2).$$

Also, Since $\lambda V_1(x_o) = \lambda z_1$,

$$U(\lambda z_1) = \Phi(\lambda V_1)(y_0) = \lambda\Phi(V_1)(y_0) = \lambda U(z_1). \quad \blacksquare$$

$$V_0(x) = f_0(x)x_0, \quad W_0(y) = \Phi(V_0)(y) = g_0(y)y_0.$$

$$U(z) = \Phi(V)(y_0) \text{ for } V \text{ such that } V(x_0) = z.$$

Give $z \in X$ let $V \in \mathcal{A}$ with $V(x_0) = z$, $\|V\| = \|z\|$. Then,

$$\|U(z)\| = \|\Phi(V)(y_0)\| \le \|\Phi\|\|V\| = \|\Phi\|\|z\|. \quad \blacksquare$$

Proof of the Theorem f. boundedness of U^{-1}

$$V_0(x) = f_0(x)x_0, \quad W_0(y) = \Phi(V_0)(y) = g_0(y)y_0.$$

$$U(z) = \Phi(V)(y_0) \text{ for } V \text{ such that } V(x_0) = z.$$

Following c. Let $y \in Y$ and let W be a rank one operator in $\mathcal{B}$ such that $W(y_0) = y, \quad \|W\| = \|y\|$. Let $V \in \mathcal{A}$ be such that $\Phi(V) = W$ and let $z = V(x_0)$. Then,

$$U(z) = \Phi(V)(y_0) = W(y_0) = y.$$

$$\|U^{-1}(y)\| = \|z\| = \|\Phi^{-1}(W)(x_0)\| \leq \|\Phi^{-1}\|\|W\| = \|\Phi^{-1}\|\|y\|. \quad \blacksquare$$

$$V_0(x) = f_0(x)x_0, \quad W_0(y) = \Phi(V_0)(y) = g_0(y)y_0.$$

$U(z) = \Phi(V)(y_0)$ for V such that $V(x_0) = z$.

Let $z \in X$ and $A \in \mathcal{A}$. Let $V \in \mathcal{A}$ such that $V(x_o) = z$. Then, $AV(x_0) = A(z)$ so,

$$UA(z) = U(A(z)) = \Phi(AV)(y_0) = \Phi(A)\Phi(V)(y_0) = \Phi(A)U(z).$$

It follows that

$$UA = \Phi(A)U,$$

or,

$$\Phi(A) = UAU^{-1}. \quad \blacksquare$$

Another theorem

Theorem. (JPS)

Let X and Y be Banach spaces and let $\mathcal{A}$ be a Banach subalgebra of $L(X)$ containing $F(X)$. Let

$$\Phi : \mathcal{A} \to L(Y)$$

be injective bounded homomorphism.
Then, there is an isomorphism $U : X \twoheadrightarrow Y_0 \subseteq Y$ such that for all $A \in \mathcal{A}$

$$UA = \Phi(A)U.$$

Moreover, for all $x \in X$,

$$\|\Phi\|^{-1}\|x\| \leq \|Ux\| \leq \|\Phi\|\|x\|.$$

Also, if X is complemented in X^{**}, Y_0 is complemented in Y.

Proposition.

Let X and Y be Banach spaces and let $\Phi : \overline{F(X)} \to L(Y)$ be injective bounded homomorphism. Then there are $U \in L(X, Y)$ and $W \in L(Y, X^{**})$ such that WV is the natural injection of X into X^{**} and $\|U\|, \|W\| \le \|\Phi\|$.

Proof: Let $V_0 = f_0 \otimes x_0$ be a norm one projection in $L(X)$ ($\|f_0\| = \|x_0\| = f_0(x_0) = 1$). For $z \in X$ and $h \in X^*$ define A_z and B_h by

$$A_z(x) = f_0(x)z, \quad B_h(x) = h(x)x_0.$$

Note that A_z and B_h are rank one operators, that $\|A_z\| = \|z\|$, $\|B_h\| = \|h\|$, and that $z \to A_z$ and $h \to B_h$ are continuous linear operations. Also, $A_{x_0} = B_{f_0} = V_0$.

Proposition

For all $z \in X, h \in X^*$ and all $x \in X$,

$$B_h A_z(x) = f_0(x) B_h(z) = f_0(x) h(z) x_0 = h(z) A_{x_0}(x),$$

i.e., $B_h A_z = h(z) A_{x_0}$.
Fix $y_0 \in \mathrm{Ran}(\Phi(V_0))$ and $g_0 \in Y^*$ with $\|g_0\| = \|y_0\| = g_0(y_0) = 1$ and Define $U : X \to Y$ by

$$U(z) = \Phi(A_z)(y_0), \quad \text{for} \quad z \in X.$$

Define also $W : Y \to X^{**}$ by

$$W(y)(h) = g_0(\Phi(B_h)(y)), \quad \text{for} \quad h \in X^*.$$

Note, $\|U\|, \|W\| \leq \|\Phi\|$.

Proposition

Then, for all $z \in X, h \in X^*$,

$$\begin{aligned}(WU(z))(h) &= g_0(\Phi(B_h)(U(z)) = g_0(\Phi(B_h)\Phi(A_z)(y_0)) \\ &= g_0(\Phi(B_h A_z)(y_0)) = g_0(h(z)\Phi(A_{x_0})(y_0)) \\ &= h(z)g_0(\Phi(V_0)(y_0)) = h(z),\end{aligned}$$

i.e.,

$$WU(z) = i(z)$$

where $i : X \to X^{**}$ is the natural injection. ■

Proof of "Another theorem"

Theorem. (JPS)

Let X and Y be Banach spaces and let $\mathcal{A}$ be a Banach subalgebra of $L(X)$ containing $F(X)$. Let

$$\Phi : \mathcal{A} \to L(Y)$$

be injective bounded homomorphism.
Then, there is an isomorphism $U : X \twoheadrightarrow Y_0 \subseteq Y$ such that for all $A \in \mathcal{A}$

$$UA = \Phi(A)U.$$

Moreover, for all $x \in X$,

$$\|\Phi\|^{-1}\|x\| \leq \|Ux\| \leq \|\Phi\|\|x\|.$$

Also, if X is complemented in X^{**}, Y_0 is complemented in Y.

Proof of "Another theorem"

Proof: Define U and W as in the proof of the Proposition. If $C \in \mathcal{A}$ and $z \in X$ then

$$\begin{aligned} UC(z) &= \Phi(A_{Cz})(y_0) = \Phi(CA_z)(y_0) \\ &= \Phi(C)\Phi(A_z)(y_0) = \Phi(C)U(z). \end{aligned}$$

So,

$$UC = \Phi(C)U.$$

We already showed that $\|Ux\| \leq \|\Phi\|\|x\|$. To prove the lower bound, for all $z \in X$,

$$\|\Phi\|\|Uz\| \geq \|W\|\|Uz\| \geq \|WUz\| = \|z\|.$$

Finally, if P is a projection from X^{**} onto X (identified with iX), then UPW is a projection from Y onto UX. ■

Remarks

- **Open:** How many closed ideals are there in $L(L_p)$, $p \neq 2$, up to Banach space isomorphism?

- Automatic continuity: The main theorem and its corollaries can be strengthen. The continuity of Φ and Φ^{-1} is automatic and need not be assumed.

Theorem. (JPS)

Let $\mathcal{A}$ be a Banach subalgebra of $L(X)$ and $\mathcal{B}$ a Banach subalgebra of $L(Y)$. Assume that $\mathcal{A} \supseteq F(X)$ (the finite rank operators) and that $\mathcal{B} \supseteq F(Y)$. Assume that

$$\Phi : \mathcal{A} \twoheadrightarrow \mathcal{B}$$

is injective and surjective homomorphism.Then, there is an isomorphism $U : X \twoheadrightarrow Y$ such that for all $A \in \mathcal{A}$

$$\Phi(A) = UAU^{-1}.$$

Proof

Proof: Assume first that $I_X \in \mathcal{A}$. Then, since $\Phi(I_X)$ commutes with all finite rank operators, $\Phi(I_X) = I_Y$. Assume that there are $A_n \in \mathcal{A}$, with $\|A_n\| = 1$ and $\|\Phi(A_n)\| \to \infty$. By Banach–Steinhaus, there is a $y_0 \in Y$ such that $\|\Phi(A_n)(y_0)\| \to \infty$ and an $g_0 \in Y^*$ such that $g_0(\Phi(A_n)(y_0)) \to \infty$.
Let $A \in \mathcal{A}$ be such that $\Phi(A) = g_0 \otimes y_0$. Since $\{AA_n\}$ is a bounded sequence there is a $\lambda_0 > 0$ such that, if $|\lambda| \leq \lambda_0$, $I_X - \lambda AA_n$ are all invertible. It follows that $I_Y - \lambda\Phi(A)\Phi(A_n)$ are all invertible for $|\lambda| \leq \lambda_0$. But, $\lambda_n = (g_0(\Phi(A_n)(y_0)))^{-1} \to 0$ and

$$(I_Y - \lambda_n\Phi(A)\Phi(A_n))(y_0) = y_0 - \lambda_n g_0(\Phi(A_n)(y_0))y_0 = 0.$$

A contradiction. So Φ is bounded. Similarly, Φ^{-1} is bounded.
If $I_X \notin \mathcal{A}$ (and then necessarily $I_Y \notin \mathcal{B}$) extend Φ naturally to the algebra generated by $\mathcal{A}$ and I_X. ■

The End