Differential Geometry, homework assignment no. 1

Due by November 26 at 3PM

If you are confident that you can present the solution to a certain question on the blackboard in class, then you can write a very short answer (e.g. "It follows directly from the definitions").

1. Let $f = (f_1, \dots, f_k) : M \to \mathbb{R}^k$ be a submersion. Prove that at point $p \in M$, the differentials
$$(df_1)_p, \dots (df_k)_p : T_pM \to \mathbb{R} \ \ (= T_{f(p)}\mathbb{R}),$$
which are linear maps, are linearly independent (i.e., prove that there is no non-trivial linear combination of these linear maps that vanishes identically).

2. Let M and N be smooth manifolds. Introduce a natural smooth structure that makes the Cartesian product $M \times N$ a smooth manifold.

3. Write $\mathbb{RP}^n$ for the collection of lines through the origin in $\mathbb{R}^{n+1}$. Describe a natural smooth structure on $\mathbb{RP}^n$, and prove that it is a smooth n-dimensional manifold.

4. Let $\pi : S^n \to \mathbb{RP}^n$ be the map that associates with a point $x \in S^n \subseteq \mathbb{R}^{n+1}$ the line through the origin that passes through x. Prove that π is a smooth two-to-one map, which is a local diffeomorphism.

5. Let $f : N \to M$ be an embedding map. Prove that the smooth structure on $f(N)$ induced by the ambient manifold M coincides with the smooth structure on $f(N)$ that is induced from N through the homeomorphism f.

6. Hadamard's theorem: Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth map satisfying $\text{rank} f'(x) = 2$ for all $x \in \mathbb{R}^2$ (i.e., a local diffeomorphism, it's both an immersion and a submersion). Assume that
$$\lim_{|x| \to \infty} |f(x)| = \infty$$
(equivalently, the preimage of a compact is a compact. It is called a proper map). Prove that f is a diffeomorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$.

   Hint: If you know what's a covering map, then you may use this. If not, then you may think of the line segment between two points x and y with $f(x) = f(y)$. What is its image under f? What happens to the preimage when you slowly deform this image, without moving the point $p = f(x) = f(y)$?