Differential Geometry, homework assignment no. 2

Please submit your solution in pdf format by December 24 at 3PM at the link:
`https://www.dropbox.com/request/uDpPacz23QM2TDQDFB3b`

You are asked to solve questions 1–6 and at least one out of questions 7–9. The rule that we had for "very short answers" still applies.

1. Verify the Jacobi identity: For any vector fields X, Y and Z,

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

2. (a) Let X be a vector field such that $Xf \equiv 0$ for any smooth, scalar function f. Prove that $X \equiv 0$.
   (b) Let f be a smooth, scalar function with $df \equiv 0$. Prove that f is locally-constant (i.e., constant on each connected component).

3. If $f : M \to N$ is a submersion, $t \in N$ and $x \in L = f^{-1}(t)$, then $T_x L = \ker(f_*)$.

4. Recall the charts for the cotangent bundle T^*M we constructed in class (the "associated coordinates". In Lee's book these are "natural coordinates"). Show that they form a smooth atlas for T^*M. Bonus: Prove that T^*M is diffeomorphic to TM.

5. Let $\pi : \mathbb{R}^{2n} \setminus \{0\} \to \mathbb{RP}^{2n-1}$ be the map that associates with any point the line through the origin in which it lies. Prove that there is a 1-form ω on $\mathbb{RP}^{2n-1}$ with

$$\pi^*\omega = \sum_{i=1}^{n} \frac{x_{n+i}dx_i - x_i dx_{n+i}}{|x|^2}.$$

6. Let $f : \mathbb{C} \to \mathbb{C}$ be a smooth map. We write $Re(f), Im(f) : \mathbb{C} \to \mathbb{R}$ for the real and imaginary parts. Prove that f is a local diffeo if and only if $dRe(f) \wedge dIm(f) \neq 0$, and that this condition is equivalent to $df \wedge d\bar{f} \neq 0$.

7. Let $X_1, \ldots, X_n$ be smooth vector fields in an n-dimensional manifold M with $[X_i, X_j] \equiv 0$ for all i, j. Assume that they are linearly independent at a given point $p \in M$. Define

$$f(t_1, \ldots, t_n) = \varphi_{t_1}^{X_1} \circ \varphi_{t_2}^{X_2} \ldots \varphi_{t_n}^{X_n}(p),$$

where $(\varphi_t^{X_i})_{t\in\mathbb{R}}$ is the one-paramater group of transformations associated with X_i. Prove that f is a local diffeomorphism, from a neigborhood of the origin in $\mathbb{R}^n$ to a neighborhood of p in M, with

$$f_*\left(\frac{\partial}{\partial t_i}\right) = X_i.$$

[You may assume without proof that f is smooth]

8. Let M be a manifold, and let V be a smooth vector field on M. Consider the transformations $(\varphi_t)_{t\in\mathbb{R}}$ associated with V. Prove that when M is compact, $\varphi_t : M \to M$ is a diffeomorphism for any $t \in \mathbb{R}$. [hint: Prove first that for some $\varepsilon > 0$, $\varphi_t(x)$ is well-defined for all $x \in M$ and $|t| < \varepsilon$].

9. Let C be a connected, one-dimensional manifold. Prove that C is diffeomorphic to S^1 if it is compact, and otherwise it is diffeomorphic to $(0, 1)$.
   Bonus: Read in Lee's book or elsewhere the definition of a manifold with boundary. Prove that if we allow boundary, then there are two additional options for C: It can also be diffeomorphic to $[0, 1]$ or to $[0, 1)$.