Differential Geometry, homework assignment no. 3

Please submit your solution in pdf format by January 14 at 3PM at the link:
`https://www.dropbox.com/request/qU3vn6DC1YrfT78ZF7W6`

1. Let X, Y be vector fields and let ω be a 1-form. Prove the formula
$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]).$$

2. Let $M \subseteq \mathbb{R}^3$ be a two-dimensional, embedded submanifold with the induced Riemannian metric. Let $U \subseteq \mathbb{R}^3$ be a neighborhood of zero and let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function with $f(0) = 0$ and $\nabla f(0) = 0$. Assume that
$$M \cap U = \{(x,y,z) \in U \,;\, z = f(x,y)\}.$$
   (a) Find a local orthonormal coframe ω_1, ω_2 near $0 \in M$, expressed explicitly in terms of the function f and its derivatives.
   (b) Set $\omega = \omega_1 + i\omega_2$ and find explicitly a real-valued 1-form ϕ with $d\omega = i\phi \wedge \omega$.
   (c) Recall that $d\phi = K\omega_1 \wedge \omega_2$ where the Gauss curvature K depends only on the Riemannian metric on M. Prove *Theorema Egregium*, that at the origin,
$$K = \det \nabla^2 f(0)$$

3. Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function and consider the Riemannian metric $g = e^u[(dx)^2 + (dy)^2]$. Prove the formula
$$K = -\frac{1}{2} e^{-u} \Delta u$$

4. Verify that $K \equiv -1$ for $\mathbb{H}^2$ (the metric $\frac{(dx)^2+(dy)^2}{y^2}$ in the upper half-plane), that $K \equiv 1$ for the unit sphere S^2, and that $K \equiv 0$ for the torus T^2 (with which metric?).

5. Use partitions of unity in order to show that any smooth manifold M admits a Riemannian metric. Conclude that TM is diffeomorphic to T^*M.

6. Let X be a vector field on a manifold M and write $(\varphi_t)_{t\in\mathbb{R}}$ for the associated one-parameter group of transformations. Let ω be a 1-form, and consider the Lie derivative
$$L_X\omega := \left.\frac{d}{dt}\varphi_t^*\omega\right|_{t=0},$$
which is a 1-form on M. Recall the formula for the Lie derivative of a vector field, and use it to show that for any vector field Y,
$$L_X\omega(Y) = d\omega(X,Y) + Y\omega(X).$$