Differential Geometry, homework assignment no. 4

Please submit your solution in pdf format by February 4 at 3PM at the link:
`https://www.dropbox.com/request/zBkX8w3X9OUSlcRPIqmC`

You are asked to solve all four questions on this page and at least two out of the six questions on the next page. The rule we had for "short answers" still applies.

1. A top form ω on a Riemannian manifold M is called a Riemannian volume form if $\omega(e_1, \ldots, e_n) = \pm 1$ for any orthonormal basis $e_1, \ldots, e_n \in T_pM$. Let $M \subseteq \mathbb{R}^n$ be a hypersurface equipped with the induced Riemannian structure, with unit normal N. Prove that $i_N\omega$ is a Riemannian volume form on M.

2. We work in $\mathbb{R}^4$ with coordinates t, x_1, x_2, x_3. Suppose that

$$F = \sum_{i=1}^{3} E_i dx_i \wedge dt + \sum_{i=1}^{3} B_i dx_{i+1} \wedge dx_{i+2}$$

where $x_{i+3} = x_i$ and where E_i, B_i are scalar functions on $\mathbb{R}^4$. Verify that $dF = 0$ iff we have the two homogeneous *Maxwell equations* (in certain units)

$$div(B) = 0 \qquad \text{and} \qquad curl(E) + \frac{\partial B}{\partial t} = 0,$$

where div and $curl$ act in the x-variables. Define the Hodge $*$ operator as a linear operator on 2-forms via $*(dx_{i+1} \wedge dx_{i+2}) = dt \wedge dx_i$ and $*(dx_i \wedge dt) = dx_{i+1} \wedge dx_{i+2}$. Set $j = d(*F)$. Identify the components J_1, J_2, J_3, ρ of the 3-form j in $\mathbb{R}^4$ so that

$$div(E) = \rho \qquad \text{and} \qquad curl(B) - \frac{\partial E}{\partial t} = J.$$

3. Use Stokes formula to prove *Cauchy's integral formula*: For any bounded, open set $\Omega \subseteq \mathbb{C}$ with a smooth boundary and for any smooth function $f : \overline{\Omega} \to \mathbb{C}$,

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{z - z_0} dz - \frac{1}{\pi} \int_{\Omega} \frac{\partial f / \partial \bar{z}}{z - z_0} dx \wedge dy \qquad (z_0 \in \Omega).$$

Hint: Consider a small disc around z_0 whose radius ε will later tend to zero.

4. Let $f : \mathbb{R}^n \to \mathbb{R}^k$ be a smooth map with $k < n$ and set $M = \{x \in \mathbb{R}^n \,;\, f(x) = 0\}$. Assume that $M \neq \emptyset$ and that zero is a regular value of f. Prove that M is orientable.

5. Recall the n-torus $\mathbb{T}^n = (S^1)^n$. Prove that

$$\dim H^k(\mathbb{T}^n) \geq \binom{n}{k}. \tag{1}$$

Hint: Look at integrals on that many sub-torii $(S^1)^k$. Bonus: Show that equality holds in equation (1).

6. Let X be a vector field on a manifold M and write $(\varphi_t)_{t\in\mathbb{R}}$ for the associated one-parameter group of transformations. Let ω be a k-form, and consider the Lie derivative

$$L_X\omega := \frac{d}{dt}\varphi_t^*\omega\bigg|_{t=0},$$

which is a k-form on M. Prove Cartan's formula

$$L_X\omega = i_X(d\omega) + d(i_X\omega).$$

Hint: Maybe use the case of 1-forms, and think on the effect of wedge products.

7. De Rham cohomology with compact support: Let ω be a compactly-supported top form on $\mathbb{R}^n$ with $\int_{\mathbb{R}^n}\omega = 0$. Prove that there exists a *compactly-supported* $(n-1)$-form α with $\omega = d\alpha$.

*8. Prove that a smooth, odd function $f : S^n \to S^n$ has an odd degree.
Hint: Maybe induction on the dimension.

9. Let $\Omega \subseteq \mathbb{C}$ be a bounded, open set and let $f : \overline{\Omega} \to \mathbb{C}$ be a smooth function. Denote $M = \{(w, z) \in \mathbb{C}^2\,;\, z \in \Omega, w = f(z)\}$. Prove *Wirtinger's inequality*:

$$\frac{i}{2}\int_M (dz \wedge d\bar{z} + dw \wedge d\bar{w}) \leq Area(M)$$

with equality if and only if f is holomorphic.

10. We work in $\mathbb{R}^{6n}$, and use $q_i = (q_{i,j})_{j=1,2,3} \in \mathbb{R}^3$ and $p_i = (p_{i,j})_{j=1,2,3} \in \mathbb{R}^3$ as coordinates, with $i = 1, \ldots, n$. Consider the symplectic form

$$\omega = \sum_{i=1}^{n}\sum_{j=1}^{3} dp_{i,j} \wedge dq_{i,j}$$

and the Hamiltonian

$$H = \sum_{i=1}^{n}\frac{|p_i|^2}{2} - \sum_{i\neq k}\frac{1}{|q_i - q_k|}.$$

Prove that the Hamiltonian flow of this system corresponds to Newton's laws of motion of n unit point masses in space. Bonus: Verify Noether's theorem here for the case of translational symmetries and the case of rotational symmetries.