Introduction to Riemann Surfaces, exercise sheet no. 1

1. Let $z_0 \in \mathbb{C}$ and let ϕ be holomorphic near z_0. Suppose that $\{(U_t, \phi_t)\}_{t\in[a,b]}$ is an analytic continuation of ϕ along $\gamma : [a, b] \to \mathbb{C}$ with $\gamma(a) = z_0$. Prove that the function $t \mapsto \phi_t(\gamma(t))$ is uniquely determined.

2. Prove that any branch of z^α, with $\alpha \in \mathbb{R}$, has an analytic continuation along any curve γ that does not pass through the origin.

3. Let $z_0, z_1, \dots, z_n \in \mathbb{C}$ be fixed. Let $\phi(z)$ be a holomorphic function near z_0 with $\phi^m(z) \equiv \prod_{i=1}^n (z - z_i)$. Let $\gamma : [a, b] \to \mathbb{C}$ be a loop about z_0 that avoids the points $z_1, \dots, z_n$, and assume that (U_t, ϕ_t) is an analytic continuation of ϕ along γ. Prove that
$$\frac{\phi_b(z_0)}{\phi_a(z_0)} = \exp\left(\frac{2\pi i}{m} \sum_{j=1}^n \mathrm{ind}(\gamma, z_j)\right).$$

4. Let $f : D \to \mathbb{C}$ be an invertible holomorphic map whose image is convex, where $D = D(0, 1) \subseteq \mathbb{C}$ is the unit disc. Prove Carathéodory's inequality
$$|f''(0)| \le 2|f'(0)|$$
(hint: Use Schwartz lemma for $g(z) = f^{-1}\left(\frac{f(\sqrt{z})+f(-\sqrt{z})}{2}\right)$. The Schwartz lemma states that a holomorphic $f : D \to D$ with $f(0) = 0$ satisfies $|f'(0)| \le 1$).

5. Let $P, Q : D(0, R) \to \mathbb{C}$ be holomorphic functions. Prove that there is a holomorphic solution in $D(0, R)$ to the differential equation
$$u'' + Pu' + Qu = 0. \tag{1}$$
Moreover, prove that the space of such solutions V is a two-dimensional linear space. (one possible hint: Write $P(z) = \sum_n p_n z^n, Q(z) = \sum_n q_n z^n$ with $\limsup |a_n|^{1/n} \le 1/R$, and look for a solution of the form $u(z) = \sum_{n=0}^\infty u_n z^n$.)

6. Suppose P, Q are holo. in an open set $U \subseteq \mathbb{C}$ and $\gamma : [0, 1] \to U$. Prove that any solution of (1) near the point $\gamma(0)$ has an analytic continuation along γ, which is still a solution.

7. From the previous exercise we learn that given a loop $\gamma : [0, 1] \to U$ with $\gamma(0) = z_0$, we may consider the space V of holo. solutions near z_0. Then we have a monodromy map: $V \ni u \mapsto M_\gamma u \in V$. It is a linear map.

   If $P(z) = A/z$ and $Q(z) = B/z^2$, and $\alpha_1, \alpha_2 \in \mathbb{C}$ are two distinct solutions of $\alpha(\alpha - 1) + A\alpha + B = 0$, then the monodromy map of a simple loop around zero has eigenvalues $e^{2\pi i\alpha_1}$ and $e^{2\pi i\alpha_2}$. Hint: Look for solutions of the form z^α.