1 Solution one

Suppose that $A \subset S^{n-1}$ is a Borel set. Define

$$d(x,A) = \inf_{y \in A} d(x,y)$$

where $\cos d(x,y) = x \cdot y$ for $x, y \in S^{n-1}$. We write σ for the standard surface area measure on S^{n-1}, normalized to be a probability measure.

Lemma 1.1. *For any $t > 0$, the set*

$$L = \{x \in S^{n-1}\,;\, d(x,A) = t\}$$

has zero σ-measure.

Proof. Let $x_0 \in L$ and fix $0 < r < t$. We claim that

$$\min_{y; d(y,x_0)=r} d(x,A) = t - r. \tag{1}$$

Indeed, the function $x \mapsto d(x,A)$ is 1-Lipschitz, hence the minimum on the left-hand side of (1) is attained. By the 1-Lipschitzness,

$$\inf_{y;d(y,x_0)=r} d(x,A) \geq d(x_0,A) - r = t - r.$$

Now, for any $\varepsilon > 0$ there is $z \in A$ with $d(x_0, z) < t + \varepsilon$. The geodesic from z to x_0 must intersect the sphere $\{y \in S^{n-1}\,;\, d(y,x_0) = r\}$, and at the point of the intersection, the distance to A is at most $t - r + \varepsilon$. Therefore $\inf_{y;d(y,x_0)=r} d(x,A) \leq t - r$, and (1) is proven.

From (1) we learn the following: For any $x_0 \in L$, $0 < r < t$ there exists x with $d(x, x_0) = r$ such that $d(x,A) = t - r$ and hence

$$B(x,r) \cap L = \emptyset$$

where $B(x_0, r) = \{x \in S^{n-1}\,;\, |x - x_0| < r\}$. This implies that there is a ball of radius $r/2$ in $B(x_0,r)$ which is disjoint from L. Therefore, for any $x_0 \in L$, $0 < r < t$,

$$\sigma(L \cap B(x_0,r)) \leq (1 - c_n)\sigma(B(x_0,r)) \tag{2}$$

where $c_n > 0$ is a number depending only on the dimension (in fact, $c_n = 2^{-(n-1)}$ works).

However, suppose that $\sigma(L) > 0$. By the Lebesgue density theorem, there is a density point $x_0 \in L$. That is,

$$\lim_{\varepsilon \to 0^+} \frac{\sigma(L \cap B(x_0,\varepsilon))}{\sigma(B(x_0,\varepsilon))} = 1,$$

in contradiction to (2). □

Note that $d(x,y) = t$ if and only if $|x - y| = \sqrt{2 - 2\cos t}$. Therefore also the set

$$L = \{x \in S^{n-1}\,;\, \inf_{y \in A} |x - y| = t\} = \{x \in S^{n-1}\,;\, d(x,A) = \arccos(1 - t^2/2)\}$$

is of zero σ-measure.

2 Solution two – An easy way out

We could have defined

$$A_\varepsilon = \{x \in S^{n-1}\,;\, d(x, A) < \varepsilon\}.$$

Then we proved that the function $A \mapsto \sigma(A_\varepsilon)$ is lower semi-continuous with respect to the Hausdorff metric, that is,

$$A_m \longrightarrow A \qquad \Longrightarrow \qquad \liminf_{m\to\infty} \sigma(A_m) \le \sigma(A).$$

A lower-semi continuous function admits a minimum on a compact set, and since

$$\mathcal{F} = \{A \in Closed(S^{n-1})\,;\, \sigma(A) \ge 1/2\}$$

is compact, then $\min_{A\in\mathcal{F}} \sigma(A_\varepsilon)$ is attained, and the set of minimizers is compact.