Rigidity of Riemannian embeddings of discrete metric spaces

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Abstract

Let M be a complete, connected Riemannian surface and suppose that $\mathcal{S} \subset M$ is a discrete subset. What can we learn about M from the knowledge of all distances in the surface between pairs of points of $\mathcal{S}$? We prove that if the distances in $\mathcal{S}$ correspond to the distances in a 2-dimensional lattice, or more generally in an arbitrary net in $\mathbb{R}^2$, then M is isometric to the Euclidean plane. We thus find that Riemannian embeddings of certain discrete metric spaces are rather rigid. A corollary is that a subset of $\mathbb{Z}^3$ that strictly contains $\mathbb{Z}^2 \times \{0\}$ cannot be isometrically embedded in any complete Riemannian surface.

1 Introduction

The collection of distances between pairs of points in a fine net in a Riemannian manifold M provides information on the geometry of the underlying manifold. A common theme in the mathematical literature is that the geometric information on M that one extracts from a discrete net is *approximate*. As the net gets finer, it better approximates the manifold. Unless one makes substantial assumptions about the manifold M, knowledge of all distances in the net typically implies that various geometric parameters of M can be estimated to a certain accuracy.

The question that we address in this paper is slightly different: Is it possible to obtain *exact* geometric information on the manifold M from knowledge of the distances between pairs of points in a discrete subset of M? We show that the answer is sometimes affirmative.

Recall that a discrete set $L \subseteq \mathbb{R}^n$ is a net if there exists $\delta > 0$ such that $d(x, L) < \delta$ for any $x \in \mathbb{R}^n$. Here, $d(x, L) = \inf_{y \in L} |x - y|$ and $|x| = \sqrt{\sum_i x_i^2}$ for $x \in \mathbb{R}^n$. For example, any n-dimensional lattice in $\mathbb{R}^n$ is a net. We say that L embeds isometrically in a Riemannian manifold M if there exists $\iota : L \to M$ such that for all $x, y \in L$,

$$d(\iota(x), \iota(y)) = |x - y|,$$

where d is the Riemannian distance function in M. We prove the following:

Theorem 1.1. *Let M be a complete, connected, 2-dimensional Riemannian manifold. Suppose that there exists a net in $\mathbb{R}^2$ that embeds isometrically in M. Then the manifold M is flat and it is isometric to the Euclidean plane.*

The conclusion of Theorem 1.1 does not hold if we merely assume that M is a Finsler manifold rather than a Riemannian manifold. Indeed, we may modify the Euclidean metric on $\mathbb{R}^2$ in a disc that is disjoint from the net $L \subseteq \mathbb{R}^2$, and obtain a Finsler metric that induces the same distances among points in the complement of the disc. This is proven in Burago and Ivanov [8]. Theorem 1.1 allows us to conclude that certain discrete metric spaces embed in 3-dimensional Riemannian manifolds but not in 2-dimensional ones:

Corollary 1.2. *Let $X \subseteq \mathbb{R}^3$ be a discrete set that is not contained in any affine plane, yet there exists an affine plane $H \subset \mathbb{R}^3$ such that $X \cap H$ is a net in H. Endow X with the Euclidean metric. Then X does not embed isometrically in any 2-dimensional, complete Riemannian manifold.*

In view of Corollary 1.2 we define the *asymptotic Riemannian dimension* of a metric space as the minimal dimension of a complete Riemannian manifold in which it embeds isometrically. (It is undefined if there is no such Riemannian manifold). For example, Corollary 1.2 tells us that the asymptotic Riemannian dimension of the metric space

$$X = \left(\mathbb{Z}^2 \times \{0\}\right) \cup \{(0,0,1)\} \subseteq \mathbb{R}^3$$

is exactly 3. It seems to us that the asymptotic Riemannian dimension captures the large-scale geometry of the metric space, hence the word asymptotic. In contrast, in the case of a finite metric space, any reasonable definition of Riemannian dimension should impose topological constraints on the manifold, since any finite, non-branching metric space may be isometrically embedded in a two-dimensional surface of a sufficiently high genus. We are not yet sure whether the n-dimensional analog of Theorem 1.1 holds true. The following result is valid in any dimension:

Theorem 1.3. *Let M be a complete, connected, n-dimensional Riemannian manifold. Suppose that there exists a net in $\mathbb{R}^n$ that embeds isometrically in M. Then M is diffeomorphic to $\mathbb{R}^n$.*

In the case where the curvature tensor of M from Theorem 1.3 is assumed compactly supported, it is not too difficult to prove that M is isometric to the Euclidean space $\mathbb{R}^n$, by reducing matters to solved partial cases of the *boundary distance conjecture* of Michel [25]. This conjecture suggests that in a simple Riemannian manifold with boundary, the collection of distances between boundary points determines the Riemannian structure, up to an isometry. To date, Michel's conjecture has been proven only in two dimensions, by Pestov and Uhlmann [27].

Definition 1.4. *We say that a subset X of an n-dimensional, complete, connected Riemannian manifold M is metrically rigid, if whenever X isometrically embeds in a complete, connected, n-dimensional Riemannian manifold $\tilde{M}$, necessarily $\tilde{M}$ is isometric to M.*

Nets in the Euclidean plane are metrically rigid, and so are random instances of a Poisson process with uniform intensity in the plane, as we argue below. One interesting question in this direction is the metric rigidity of nets in complete, simply-connected Riemannian manifolds of non-positive curvature. Another natural question is whether there exist finitary versions of Theorem 1.1, in which we isometrically embed a large, finite chunk of the net L and wish to obtain some geometric corollaries.

We proceed to describe a notion slightly more inclusive than that of a net, which also covers instances of Poisson processes. We call an open set $S \subseteq \mathbb{R}^n$ a *sector* if there exist $x_0 \in \mathbb{R}^n$ and an open, connected set $U \subseteq S^{n-1} = \{x \in \mathbb{R}^n \, ; \, |x| = 1\}$ such that $S = \{x_0 + r\theta \, ; \, \theta \in U, r > 0\}$. For a function $\varphi : (0, \infty) \to (0, \infty)$ and for $n \geq 2$ we write

$$\text{Subgraph}_n(\varphi) = \{(x, y) \in \mathbb{R} \times \mathbb{R}^{n-1} \, ; \, x > 0, |y| \leq \varphi(x)\} \subset \mathbb{R}^n.$$

By a *quasi-net* we mean a discrete subset $L \subset \mathbb{R}^n$ which satisfies at least one of the following conditions:

(QN1) There exists a non-decreasing function $\varphi : (0, \infty) \to (0, \infty)$ with $\lim_{r \to \infty} \varphi(r)/\sqrt{r} = 0$ for which the following holds: For any isometry $T : \mathbb{R}^n \to \mathbb{R}^n$,

$$L \cap T(\text{Subgraph}_n(\varphi)) \neq \emptyset.$$

(QN2) For any non-empty, open sector $S \subseteq \mathbb{R}^n$ there exists a sequence $(p_m)_{m \geq 1}$ with $p_m \in L \cap S$ for all m such that

$$\lim_{m \to \infty} \frac{|p_{m+1}|}{|p_m|} = 1 \qquad \text{and} \qquad \lim_{m \to \infty} |p_m| = \infty.$$

It is clear that any net in $\mathbb{R}^n$ satisfies conditions (QN1) and (QN2). A random instance of a Poisson process with uniform intensity in $\mathbb{R}^n$ is a discrete set satisfying (QN1) and (QN2), with probability one. Hence Theorem 1.1 and Theorem 1.3 are particular cases of the following:

Theorem 1.5. *Let M be a complete, connected, 2-dimensional Riemannian manifold. Suppose that there exists a discrete set in $\mathbb{R}^2$ which satisfies conditions (QN1) and (QN2) and that embeds isometrically in M. Then the manifold M is flat and it is isometric to the Euclidean plane $\mathbb{R}^2$.*

Theorem 1.6. *Let M be a complete, connected, n-dimensional Riemannian manifold. Suppose that there exists a discrete set in $\mathbb{R}^n$ which satisfies condition (QN1) and that embeds isometrically in M. Then M is diffeomorphic to $\mathbb{R}^n$.*

The remainder of this paper is devoted almost entirely to the proofs of Theorem 1.5 and Theorem 1.6. The key step in the proof of Theorem 1.5 is to show that M has no conjugate points. This enables us to make contact with the developed mathematical literature on the rigidity of Riemannian manifolds without conjugate points under topological assumptions, under curvature assumptions or under isoperimetric assumptions. The relevant literature begins with the works of Morse and Hedlund [26] and Hopf [21], and continues with contributions by Bangert and Emmerich [3, 4], Burago and Ivanov [7], Burns and Knieper [10], Busemann [11], Croke [14, 15] and others. At the final step of the argument below we apply the equality case of the isoperimetric inequality of Bangert and Emmerich [4], whose beautiful proof is based on Hopf's method.

The mathematical literature pertaining to nets that approximate a Riemannian manifold includes the analysis by Fefferman, Ivanov, Kurylev, Lassas and Narayanan [17, 18], and the works by Fujiwara [20] and by Burago, Ivanov and Kurylev [9] on approximating the spectrum and eigenfunctions of the Laplacian via a net. These works are related to the useful idea of a *diffusion map*, as in Belkin and Niyogi [5], Coifmann and Lafon [13] and Singer [28].

All of the Riemannian manifolds below are assumed to be C^2-smooth, and all parametrizations of geodesics are by arclength. Thus a geodesic here is always of unit speed. We write that a function $f(t)$ is $o(t)$ as $t \to \infty$ if $f(t)/t$ tends to zero as $t \to \infty$.

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2 Lipschitz functions

We begin the proofs of Theorem 1.5 and Theorem 1.6 with some background on geodesics and Lipschitz functions. Our standard reference for Riemannian geometry is Cheeger and Ebin [12].

We work in a complete, connected, Riemannian manifold M with distance function d. A *minimizing geodesic* is a curve $\gamma : I \to M$, where $I \subseteq \mathbb{R}$ is an interval (i.e., a connected set), with

$$d(\gamma(t), \gamma(s)) = |t - s| \qquad \text{for all } s, t \in I.$$

As is customary, our notation does not fully distinguish between the parametrized curve $\gamma : I \to M$ and its image $\gamma(I)$ which is just a subset of M, sometimes endowed with an orientation. It should be clear from the context whether we mean a parametrized curve, or its image in M.

A curve $\gamma : I \to M$ is a geodesic if the interval I may be covered by open intervals on each of which γ is a minimizing geodesic. In the case where $I = \mathbb{R}$ we say that the geodesic γ is complete. When $I = [0, \infty)$ or $I = (0, \infty)$ we say that γ is a ray, and if $I \subseteq \mathbb{R}$ is of finite length we say that γ is a geodesic segment. Since M is complete, for any $x, y \in M$ there exists a minimizing geodesic segment connecting x and y. A minimizing geodesic ray cannot intersect a minimizing geodesic segment at more than one point unless they overlap, see [12].

Let $\gamma_m : I_m \to M$ $(m = 1, 2, \ldots)$ be a sequence of geodesics. We say that the sequence converges to a geodesic $\gamma : I \to M$ if $I = \cup_{m \geq 1} \cap_{k \geq m} I_k$ and for any $t \in I$,

$$\gamma_m(t) \xrightarrow{m \to \infty} \gamma(t).$$

In the case where $I_m = I$ for all m, for any fixed $t_0 \in I$ the convergence $\gamma_m \longrightarrow \gamma$ is equivalent to the requirement that

$$\gamma_m(t_0) \xrightarrow{m \to \infty} \gamma(t_0) \qquad \text{and} \qquad \dot{\gamma}_m(t_0) \xrightarrow{m \to \infty} \dot{\gamma}(t_0).$$

Here $\dot{\gamma}(t_0) \in T_{\gamma(t_0)}M$ is the tangent vector to the geodesic γ, and T_pM is the tangent space to M at the point $p \in M$. A sequence of unparametrized geodesics is said to converge if its geodesics may be parametrized to yield a converging sequence in the above sense.

The continuity of the distance function implies that the limit of a converging sequence of minimizing geodesics, is itself a minimizing geodesic. Any sequence of geodesics passing through a fixed point $x \in M$, has a convergent subsequence. We say that a sequence of points in M tends to infinity if any compact $K \subseteq M$ contains only finitely many points from the sequence. When $x, x_1, x_2, \ldots$ are points in M with $x_m \longrightarrow \infty$ and γ_m is a minimizing geodesic connecting x with x_m, the sequence $(\gamma_m)_{m \geq 1}$ has a subsequence that converges to a minimizing geodesic ray emanating from x.

Lipschitz functions are somewhat "dual" to curves and geodesics in the following sense: Any rectifiable curve between x and y provides an upper bound for the distance $d(x, y)$. On the other hand, a 1-Lipschitz function $f : M \to \mathbb{R}$ is a function that satisfies $|f(x) - f(y)| \leq d(x, y)$ for all $x, y \in M$, and hence it provides lower bounds for the distance $d(x, y)$. When $f : M \to \mathbb{R}$ is 1-Lipschitz and $\gamma : I \to M$ is a geodesic,

$$|f(\gamma(t)) - f(\gamma(s))| \leq d(\gamma(t), \gamma(s)) \leq |t - s| \qquad \text{for all } s, t \in I. \tag{1}$$

We say that the geodesic γ is a *transport curve* of the 1-Lipschitz function f if

$$f(\gamma(t)) - f(\gamma(s)) = t - s \qquad \text{for all } s, t \in I. \tag{2}$$

Thus, the function f grows with unit speed along a transport curve. This terminology comes from the theory of optimal transport, see e.g. Evans and Gangbo [16] or [23, Section 2.1]. If $I = \mathbb{R}$ then we say that the transport curve γ is a *transport line* and if the transport curve γ is a geodesic ray then γ is called a *transport ray*. It follows from (1) and (2) that any transport curve is a minimizing geodesic.

When $\gamma : I \to M$ is a transport curve of a 1-Lipschitz function f, the function f is differentiable at $\gamma(t)$ for all t in the interior of the interval $I \subseteq \mathbb{R}$, as proven in Feldman and McCann [19, Lemma 10]. For any $t \in I$ such that f is differentiable at $\gamma(t)$, we have

$$\nabla f(\gamma(t)) = \dot{\gamma}(t). \tag{3}$$

Indeed, it follows from (2) that $\langle \nabla f(\gamma(t)), \dot{\gamma}(t) \rangle = 1$, where $\langle \cdot, \cdot \rangle$ and $|\cdot|$ are the Riemannian scalar product and norm in $T_{\gamma(t)}M$, and hence (3) follows as $|\nabla f(\gamma(t))| \leq 1$ and $|\dot{\gamma}(t)| = 1$.

Lemma 2.1. *Let $f : M \to \mathbb{R}$ be a 1-Lipschitz function. Suppose that γ_1 is a transport line of f and that γ_2 is a transport curve of f with $\gamma_1 \cap \gamma_2 \neq \emptyset$. Then $\gamma_2 \subseteq \gamma_1$. In particular, if γ_2 is a transport line as well, then the geodesics γ_1 and γ_2 coincide.*

Proof. Since both γ_1 and γ_2 are minimizing geodesics passing through a point $x \in \gamma_1 \cap \gamma_2$ in the direction $\nabla f(x)$, necessarily $\gamma_2 \subseteq \gamma_1$. ◻

It follows from Lemma 2.1 that if γ is a transport line of f and $x \in \gamma$, then for any $y \in M$,

$$y \in \gamma \qquad \Longleftrightarrow \qquad |f(y) - f(x)| = d(x, y). \tag{4}$$

The first example of a 1-Lipschitz function in M is the distance function $x \mapsto d(p,x)$ from a given point $p \in M$. Any minimizing geodesic segment connecting p to a point $y \in M$ is a transport curve of this distance function. The second example is the Busemann function of a minimizing geodesic $\gamma : [t_0, \infty) \to M$, defined as

$$B_\gamma(x) = \lim_{t \to \infty} [t - d(\gamma(t), x)] = \sup_{t \geq t_0} [t - d(\gamma(t), x)] \, . \tag{5}$$

Our definition of B_γ differs by a sign from the convention in Ballman, Gromov and Schroeder [2] and in Busemann [11]. It is well-known that the limit in (5) always exists, since $t - d(\gamma(t), x)$ is non-decreasing in t and bounded from above by $d(\gamma(t_0), x) + |t_0|$. Moreover, the function $B_\gamma : M \to \mathbb{R}$ is a 1-Lipschitz function. Thanks to our sign convention, any minimizing geodesic $\gamma : [t_0, \infty) \to M$ is a transport ray of the 1-Lipschitz function B_γ.

If a sequence of 1-Lipschitz functions $f_m : M \to \mathbb{R}$ $(m \geq 1)$ converges pointwise as $m \to \infty$ to a limit function $f : M \to \mathbb{R}$, then f is 1-Lipschitz. Moreover, the convergence is locally uniform by the Arzela-Ascoli theorem. We will frequently use the following fact: For a continuous f, the convergence $f_m \longrightarrow f$ is locally uniform if and only if whenever $M \ni x_m \longrightarrow x$, also $f_m(x_m) \longrightarrow f(x)$.

Lemma 2.2. *Let $f_m : M \to \mathbb{R}$ be a sequence of 1-Lipschitz functions that converges pointwise to $f : M \to \mathbb{R}$, and let $\gamma_m : I_m \to M$ be a sequence of geodesics converging to $\gamma : I \to M$ such that γ_m is a transport curve of f_m for all m. Then γ is a transport curve of f.*

Proof. Let $s, t \in I$. Since γ_m is a transport curve of f_m, for a sufficiently large m,

$$f_m(\gamma_m(t)) - f_m(\gamma_m(s)) = t - s. \tag{6}$$

Since the convergence $f_m \longrightarrow f$ is locally uniform in M, we have $f_m(\gamma_m(t)) \longrightarrow f(\gamma(t))$ for all $t \in I$. Letting $m \to \infty$ in (6) yields $f(\gamma(t)) - f(\gamma(s)) = t - s$. ☐

We say that a 1-Lipschitz function $f : M \to \mathbb{R}$ *induces a foliation by transport lines*, or in short *foliates*, if for any $x \in M$ there exists a transport line of f that contains x. By Lemma 2.1, in this case M is the disjoint union of the transport lines of f. When a function f foliates, it is differentiable everywhere in M. In fact, it is known that f is a $C^{1,1}$-function in this case, see [23, Theorem 2.1.13].

The following proposition describes a way to produce 1-Lipschitz functions that foliate. For $x \in M$, we denote the cut-locus of x by $\mathrm{cut}(x) \subseteq M$. See [12] for information about the cut-locus.

Proposition 2.3. *Let $(y_m)_{m \geq 1}$ be a sequence of points in M tending to infinity and let $(C_m)_{m \geq 1}$ be real numbers. Denote*

$$f_m(x) = C_m - d(x, y_m) \qquad\qquad (x \in M, m \geq 1)$$

and assume that $f_m \longrightarrow f$ pointwise in M as $m \to \infty$. Then:

(i) If cut$(y_m) = \emptyset$ for all m, then f foliates.

(ii) Suppose that f foliates and fix $x \in M$. Then for any convergent sequence of minimizing geodesics $\gamma_m : [0, d(x, y_m)] \to M$ with $\gamma_m(0) = x$ and $\gamma_m(d(x, y_m)) = y_m$,

$$\dot{\gamma}_m(0) \xrightarrow{m\to\infty} \nabla f(x).$$

Proof. Fix $x \in M$ and set $r_m = d(x, y_m)$. Since M is complete and $y_m \longrightarrow \infty$, necessarily $r_m \longrightarrow \infty$. We proceed with the proof of (ii).

(ii) Define

$$\gamma = \lim_{m\to\infty} \gamma_m.$$

For any $t > 0$, we know that $t \in [0, r_m]$ for a sufficiently large m, which implies that γ is defined on $[0, \infty)$. The geodesic γ_m is a transport curve of f_m for any m, since for $t, s \in [0, r_m]$ we have

$$f_m(\gamma_m(t)) - f_m(\gamma_m(s)) = d(\gamma_m(s), y_m) - d(\gamma_m(t), y_m) = t - s.$$

Lemma 2.2 shows that γ is a transport curve of f. Since f foliates, it is differentiable at x, and therefore $\dot{\gamma}_m(0) \longrightarrow \dot{\gamma}(0) = \nabla f(x)$ where we used (3) in the last passage.

(i) In order to show that f foliates, we need to find a transport line of f that passes through x. Since cut$(y_m) = \emptyset$, we may write $\gamma_m : (-\infty, r_m] \to M$ for the unique minimizing geodesic with

$$\gamma_m(0) = x \qquad \text{and} \qquad \gamma_m(r_m) = y_m.$$

Passing to a subsequence, we may assume that $\gamma_m \longrightarrow \gamma$ for a minimimizing geodesic γ with $\gamma(0) = x$. Since $r_m \longrightarrow \infty$, the geodesic γ is complete. The geodesic γ_m is a transport curve of f_m for any m, and from Lemma 2.2 we conclude that γ is a transport line of f that passes through x. Hence f foliates.

□

Lemma 2.4. *Let V be a metric space, and assume that with any $v \in V$ we associate a 1-Lipschitz function $f_v : M \to \mathbb{R}$. Suppose that f_v foliates for any $v \in V$, and that $f_v(x)$ varies continuously with $v \in V$ for any fixed $x \in M$. Then the map*

$$(x, v) \mapsto \nabla f_v(x)$$

is continuous in $M \times V$.

Proof. It suffices to show that for any sequence $M \times V \ni (x_m, v_m) \longrightarrow (x, v) \in M \times V$, the sequence $(\nabla f_{v_m}(x_m))_{m \geq 1}$ has a subsequence converging to $\nabla f_v(x)$. Abbreviate $f_m = f_{v_m}$ and $f = f_v$, so that $f_m \longrightarrow f$ locally uniformly by the Arzela-Ascoli theorem. For each m consider the transport line $\gamma_m : \mathbb{R} \to M$ of f_m which satisfies

$$\gamma_m(0) = x_m \qquad \text{and} \qquad \dot{\gamma}_m(0) = \nabla f_m(x_m).$$

Since $\gamma_m(0) \longrightarrow x$ as $m \to \infty$, we may pass to a convergent subsequence, and assume that $\gamma_m \longrightarrow \gamma$ for some minimizing geodesic $\gamma : \mathbb{R} \to M$ with

$$\gamma(0) = x \qquad \text{and} \qquad \dot{\gamma}(0) = \lim_{m \to \infty} \dot{\gamma}_m(0) = \lim_{m \to \infty} \nabla f_m(x_m).$$

Lemma 2.2 states that γ is a transport line of f. In particular $\dot{\gamma}(0) = \nabla f(x)$ by (3). □

3 Directional drift to infinity

From now on and until the end of Section 6, our standing assumptions are the assumptions of Theorem 1.6. We thus work in a complete, connected, n-dimensional Riemannian manifold M, with $n \geq 2$. We assume that $\iota : L \to M$ is an isometric embedding for the discrete set

$$L \subseteq \mathbb{R}^n,$$

that satisfies condition (QN1). Translating the discrete set L does not alter the validity of condition (QN1) or condition (QN2), hence we may translate L and assume for convenience that

$$0 \in L.$$

For ease of reading, and with a slight abuse of notation, we identify between a point $a \in L \subseteq \mathbb{R}^n$ and its image $\iota(a) \in M$. Thus we think of L as a subset of M, and the assumption that ι is an isometric embedding translates to

$$d(x, y) = |x - y| \qquad\qquad \text{for all } x, y \in L. \tag{7}$$

Note that for $p \in L \subseteq M$, we may speak of the Euclidean norm $|p| = \sqrt{\sum_i p_i^2} = d(0, p)$ and of the scalar product $\langle p, v \rangle = \sum_i p_i v_i$ for $v \in \mathbb{R}^n$. Given $p \in L$ we write $d_p : M \to \mathbb{R}$ for the function

$$d_p(x) = d(p, 0) - d(p, x)$$

which is a 1-Lipschitz function that vanishes at $0 \in L \subseteq M$. The following notion is in the spirit of the "ideal boundary" of a Hadamard manifold (see, e.g., [2]).

Definition 3.1. *Let $v \in S^{n-1}$ and let $B : M \to \mathbb{R}$ be a 1-Lipschitz function. We write that $B \in \partial_v M$ if*

$$B(p) = \langle p, v \rangle \qquad\qquad \textit{for all } p \in L.$$

We say that a sequence of points $p_m \in \mathbb{R}^n \ \ (m = 1, 2, \ldots)$ is *drifting in the direction of* $v \in S^{n-1}$, and we write $p_m \rightsquigarrow v$, if

$$p_m \xrightarrow{m\to\infty} \infty \qquad \text{and} \qquad \frac{p_m}{|p_m|} \xrightarrow{m\to\infty} v. \tag{8}$$

Proposition 3.2. *Let $v \in S^{n-1}$, $B : M \to \mathbb{R}$, and let $p_m \in L$ satisfy $p_m \rightsquigarrow v$. Assume that $d_{p_m} \longrightarrow B$ pointwise as $m \to \infty$. Then $B \in \partial_v M$.*

Proof. The function B is 1-Lipschitz, being the pointwise limit of a sequence of 1-Lipschitz functions. By (7), for any $q \in L$,

$$B(q) = \lim_{m\to\infty} d_{p_m}(q) = \lim_{m\to\infty} [d(p_m, 0) - d(p_m, q)] = \lim_{m\to\infty} [|p_m| - |p_m - q|] = \langle q, v\rangle,$$

where the last passage is an exercise in Euclidean geometry. Thus $B \in \partial_v M$. □

Let $p_m \in \mathbb{R}^n \ \ (m = 1, 2, \ldots)$ be a sequence of points and let $v \in S^{n-1}$. We say that $p_m \rightsquigarrow v$ *narrowly* if

$$|p_m| - \langle p_m, v\rangle \xrightarrow{m\to\infty} 0 \qquad \text{and} \qquad p_m \xrightarrow{m\to\infty} \infty. \tag{9}$$

Clearly (9) implies (8). Two properties of narrow drift are summarized in the following:

Lemma 3.3. *For any $v \in S^{n-1}$ there exists a sequence $(p_m)_{m\geq 1}$ in L with $p_m \rightsquigarrow v$ narrowly. Moreover, for any such sequence and for any $p \in L$, also $p_m - p \rightsquigarrow v$ narrowly.*

Proof. Let $(p_m)_{m\geq 1}$ be a sequence in L with $|p_m| \longrightarrow \infty$, and write $r_m = |p_m|$. Since for any $v \in S^{n-1}$,

$$\frac{|p_m - r_m v|}{\sqrt{r_m}} = \sqrt{2(r_m - \langle p_m, v\rangle)},$$

condition (9) is equivalent to

$$\frac{|p_m - r_m v|}{\sqrt{r_m}} \xrightarrow{m\to\infty} 0. \tag{10}$$

We thus need to find $p_m \in L$ with $p_m \longrightarrow \infty$ such that (10) holds true. Since $L \subseteq \mathbb{R}^n$ satisfies (QN1), there exists a non-decreasing function $\varphi : (0, \infty) \to (0, \infty)$ with $\varphi(r) = o(\sqrt{r})$ as $r \to \infty$ such that for any isometry $T : \mathbb{R}^n \to \mathbb{R}^n$,

$$L \cap T(\text{Subgraph}_n(\varphi)) \neq \emptyset. \tag{11}$$

Let $U : \mathbb{R}^n \to \mathbb{R}^n$ be a linear orthogonal transformation that maps the standard unit vector e_1 to the unit vector v, and let $T_m(x) = U(x) + m \cdot v$ be an isometry of $\mathbb{R}^n$. By applying (11) we conclude that for any $m \geq 1$ there exists a point $p_m \in L$ such that $q_m = p_m - m \cdot v$ satisfies $\langle q_m, v\rangle > 0$ and

$$|\text{Proj}_{v^\perp} q_m| \leq \varphi(\langle q_m, v\rangle), \tag{12}$$

where the linear map $\text{Proj}_{v^\perp} : \mathbb{R}^n \to \mathbb{R}^n$ is the orthogonal projection on the hyperplane orthogonal to v in $\mathbb{R}^n$. Note that $p_m \longrightarrow \infty$ since $|p_m| \geq \langle p_m, v\rangle \geq m$. As φ is non-decreasing, according to (12),

$$|\text{Proj}_{v^\perp} p_m| = |\text{Proj}_{v^\perp} q_m| \leq \varphi(\langle q_m, v\rangle) \leq \varphi(\langle p_m, v\rangle) = o(\sqrt{\langle p_m, v\rangle}) = o(\sqrt{r_m}),$$

as $r_m = |p_m|$. Since $r_m \leq |\text{Proj}_{v^\perp} p_m| + |\langle p_m, v\rangle| = \langle p_m, v\rangle + o(\sqrt{r_m})$,

$$|p_m - r_m v| \leq |\text{Proj}_{v^\perp} p_m| + |\langle p_m, v\rangle - r_m| = o(\sqrt{r_m}) + (r_m - \langle p_m, v\rangle) = o(\sqrt{r_m}),$$

proving (10). Next, given any sequence $p_m \rightsquigarrow v$ narrowly, it follows from (10) that for $q_m = p_m - p, r_m = |p_m|$ and $s_m = |q_m|$,

$$|q_m - s_m v| \leq |p_m - r_m v| + 2|p| = o(\sqrt{r_m}) + 2|p| = o(\sqrt{s_m})$$

and hence $q_m \rightsquigarrow v$ narrowly as well. □

Assumption (QN1) in Theorem 1.5 and in Theorem 1.6 may actually be replaced by assumption (QN1'), which is the condition that for any $v \in S^{n-1}$ there exists a sequence $(p_m)_{m\geq 1}$ in L with $p_m \rightsquigarrow v$ narrowly. We also note here that neither (QN1) implies (QN2) nor (QN2) implies (QN1).

Lemma 3.4. *The set $\partial_v M$ is non-empty for any $v \in S^{n-1}$. In fact, for any sequence $L \ni p_m \rightsquigarrow v$ there exists a subsequence such that $d_{p_{m_k}} \longrightarrow B$ for some $B \in \partial_v M$.*

Proof. It follows from Lemma 3.3 that there exist points $L \ni p_m \rightsquigarrow v$. The sequence $(d_{p_m})_{m\geq 1}$ consists of 1-Lipschitz functions vanishing at 0. By the Arzela-Ascoli theorem, there exists a subsequence $(d_{p_{m_k}})_{k\geq 1}$ such that $d_{p_{m_k}} \longrightarrow B$ locally uniformly for some 1-Lipschitz function $B : M \to \mathbb{R}$. By Proposition 3.2, we have that $B \in \partial_v M$. □

From Definition 3.1 it follows that for any $v \in S^{n-1}$,

$$\partial_{-v} M = -\partial_v M := \{-B\,;\, B \in \partial_v M\}. \tag{13}$$

The next proposition is the reason for introducing the notion of a narrow drift. It produces complete minimizing geodesics through points of L that interact nicely with $\partial_v M$.

Proposition 3.5. *Let $p \in L, v \in S^{n-1}$ and assume that $(p_m^+)_{m\geq 1}$ is a sequence in L with $p_m^+ \rightsquigarrow v$ narrowly, while $(p_m^-)_{m\geq 1}$ is a sequence in L satisfying $p_m^- \rightsquigarrow -v$ narrowly.*

For any m, let $\gamma_m^\pm$ be a minimizing geodesic connecting p and $p_m^\pm$. Assume that $\lim_m \gamma_m^\pm = \gamma^\pm$ for a geodesic ray $\gamma^\pm : [0,\infty) \to M$ with $\gamma^\pm(0) = p$. Then the concatenation $\gamma = \gamma^+ \cup \gamma^-$ with parametrization

$$\gamma(t) = \begin{cases} \gamma^+(t) & t \geq 0 \\ \gamma^-(-t) & t \leq 0 \end{cases} \tag{14}$$

is a transport line of B, for any $B \in \partial_v M$.

Proof. Set $r_m^\pm = |p_m^\pm - p|$. Then $r_m^\pm \longrightarrow \infty$ by (9). We parametrize our geodesics as $\gamma_m^\pm : [0, r_m^\pm] \to M$ with

$$\gamma_m^\pm(0) = p \qquad \text{and} \qquad \gamma_m^\pm(r_m^\pm) = p_m^\pm.$$

By our assumption, $\gamma_m^\pm \longrightarrow \gamma^\pm$ where $\gamma^\pm : [0,\infty) \to M$ is a geodesic with $\gamma^\pm(0) = p$. Fix $t > 0$ and $B \in \partial_v M$. In order to show that γ, as defined in (14), is a transport line of B, it suffices to show that

$$B(\gamma^+(t)) - B(\gamma^+(0)) = B(\gamma^-(0)) - B(\gamma^-(t)) = t. \tag{15}$$

Since $p_m^\pm \rightsquigarrow \pm v$ narrowly, it follows from Lemma 3.3 that $p_m^\pm - p \rightsquigarrow \pm v$ narrowly as well, i.e.

$$|p_m^\pm - p| - \langle p_m - p, \pm v\rangle \xrightarrow{m\to\infty} 0.$$

For a sufficiently large m, we know that $t \le \min\{r_m^+, r_m^-\}$. Since B is 1-Lipschitz, by (1),

$$\begin{aligned} B(\gamma_m^+(t)) - B(\gamma_m^+(0)) - t &\ge B(\gamma_m^+(r_m^+)) - B(\gamma_m^+(0)) - r_m^+ \\ &= B(p_m^+) - B(p) - r_m^+ = \langle p_m^+ - p, v\rangle - |p_m^+ - p| \xrightarrow{m\to\infty} 0. \end{aligned} \tag{16}$$

Since $-B \in \partial_{-v} M$ according to (13), arguing as in (16) we have

$$\begin{aligned} -B(\gamma_m^-(t)) + B(\gamma_m^-(0)) - t &\ge -B(\gamma_m^-(r_m^-)) + B(\gamma_m^-(0)) - r_m^- \\ &= \langle p_m^- - p, -v\rangle - |p_m^- - p| \xrightarrow{m\to\infty} 0. \end{aligned}$$

Since $\gamma_m^\pm \longrightarrow \gamma^\pm$, by taking the limit $m \to \infty$ we obtain

$$B(\gamma^+(t)) - B(\gamma^+(0)) - t \ge 0 \qquad \text{and} \qquad -B(\gamma^-(t)) + B(\gamma^-(0)) - t \ge 0.$$

The reverse inequalities are trivial by (1), and hence (15) follows. ∎

Lemma 3.6. *Let $p \in L$ and $v \in S^{n-1}$. Let $(p_m)_{m\ge1}$ and $(q_m)_{m\ge1}$ be sequences in L such that $p_m \rightsquigarrow v$ and $q_m \rightsquigarrow v$ with at least one of the drifts being narrow. Let γ_m be a minimizing geodesic from p to p_m and let η_m be a minimizing geodesic from p to q_m. Then there exists a geodesic ray γ with $\gamma(0) = p$ such that both $\gamma_m \longrightarrow \gamma$ and $\eta_m \longrightarrow \gamma$.*

Proof. Passing to convergent subsequences, we may assume that $\gamma_m \longrightarrow \gamma$ and $\eta_m \longrightarrow \eta$ for some geodesic rays γ and η with $\gamma(0) = \eta(0) = p$, and our goal is to prove that $\gamma \equiv \eta$.

Assume that the drift $p_m \rightsquigarrow v$ is narrow. By Lemma 3.4, we may pass to a subsequence, and assume that $d_{q_m} \longrightarrow B$ for a certain $B \in \partial_v M$. Since η_m is a minimizing geodesic connecting p and q_m, it is a transport curve of d_{q_m}. Since $d_{q_m} \longrightarrow B$ and $\eta_m \longrightarrow \eta$, Lemma 2.2 implies that the geodesic ray η is a transport ray of B.

According to Lemma 3.3, there exists a sequence $(\tilde{p}_m)_{m\ge1}$ with $\tilde{p}_m \rightsquigarrow -v$ narrowly. Passing to a subsequence, we may assume that $\tilde{\gamma}_m$, a minimizing geodesic from p to $\tilde{p}_m$, converges as $m \to \infty$ to a geodesic ray $\tilde{\gamma}$ with $\tilde{\gamma}(0) = p$. Recall that the drift $p_m \rightsquigarrow v$ is narrow and that

$\gamma_m \longrightarrow \gamma$. By Proposition 3.5, the concatenation $\hat{\gamma} = \gamma \cup \tilde{\gamma}$ is a transport line of B with the parametrization

$$\hat{\gamma}(t) = \begin{cases} \gamma(t) & t \geq 0 \\ \tilde{\gamma}(-t) & t \leq 0 \end{cases} \tag{17}$$

Thus $\hat{\gamma}$ is a transport line of B and η is a transport ray of B, both passing through the point p. Lemma 2.1 implies that

$$\eta \subseteq \hat{\gamma} = \gamma \cup \tilde{\gamma}. \tag{18}$$

The three curves η, γ and $\tilde{\gamma}$ are geodesic rays emanating from p. It thus follows from (18) that either $\eta = \gamma$ or else $\eta = \tilde{\gamma}$. However, $\eta : [0, \infty) \to M$ and $\gamma : [0, \infty) \to M$ are transport rays of B unlike $\tilde{\gamma} : [0, \infty) \to M$, as follows from (17), hence $\eta \equiv \gamma$. ◻

We write $S_pM = \{u \in T_pM \, ; \, |u| = 1\}$ for the unit tangent sphere at the point $p \in M$.

Proposition 3.7. *Fix $p \in L$. Then with any $v \in S^{n-1}$ there is a unique way to associate a minimizing geodesic $\gamma_{p,v} : \mathbb{R} \to M$ with $\gamma_{p,v}(0) = p$ such that the following hold:*

(i) For any $v \in S^{n-1}$, if $(p_m)_{m \geq 1}$ is a sequence in L with $p_m \rightsquigarrow v$, and γ_m is a minimizing geodesic from p to p_m, then γ_m tends to the geodesic ray $\gamma_{p,v}([0, \infty))$ as $m \to \infty$.

(ii) The map $S^{n-1} \ni v \mapsto \dot{\gamma}_{p,v}(0) \in S_pM$ is odd, continuous and onto.

(iii) For any $v \in S^{n-1}$ and $B \in \partial_v M$, the minimizing geodesic $\gamma_{p,v}$ is a transport line of B.

Proof. For $v \in S^{n-1}$ we apply Lemma 3.3 and select a sequence $(q_m)_{m \geq 1} = (q_m^{(v)})_{m \geq 1}$ in L with $q_m \rightsquigarrow v$ narrowly. Let $\eta_m = \eta_m^{(v)}$ be a minimizing geodesic segment with

$$\eta_m(0) = p \qquad \text{and} \qquad \eta_m(d(p, q_m)) = q_m. \tag{19}$$

Lemma 3.6 implies that $\eta_m \longrightarrow \eta$ for a geodesic ray $\eta : [0, \infty) \to M$ with $\eta(0) = p$. We now define

$$\gamma_{p,v}(t) = \eta(t) \qquad\qquad (t \geq 0). \tag{20}$$

Lemma 3.6 also states that whenever $L \ni p_m \rightsquigarrow v$, a minimizing geodesic γ_m from p to p_m tends to the geodesic ray $\gamma_{p,v}([0, \infty))$ as $m \to \infty$. Thus (i) holds true. The geodesic ray $\gamma_{p,v}$ is defined in (20) for all $v \in S^{n-1}$, but only for $t \geq 0$. We extend this definition by setting

$$\gamma_{p,v}(-t) := \gamma_{p,-v}(t) \qquad\qquad \text{for all } t \geq 0. \tag{21}$$

Since $q_m^{(v)} \rightsquigarrow v$ and $q_m^{(-v)} \rightsquigarrow -v$, Proposition 3.5 shows that $\gamma_{p,v} : \mathbb{R} \to M$ is a transport line of B for any $B \in \partial_v M$. Thus (iii) is proven. Since $\partial_v M \neq \emptyset$ by Lemma 3.4, the complete geodesic $\gamma_{p,v}$ is therefore minimizing. It is clear from our construction that $\gamma_{p,v}$ is uniquely determined by requirement (i).

All that remains is to prove (ii). The map $F(v) = \dot{\gamma}_{p,v}(0)$ from S^{n-1} to S_pM is odd according to (21). Let us prove its continuity. To this end, suppose that $S^{n-1} \ni v_m \longrightarrow v$, and our goal is to prove that $\gamma_{p,v_m} \longrightarrow \gamma_{p,v}$. For each fixed $m \geq 1$ we know that as $k \to \infty$,

$$q_k^{(v_m)} \longrightarrow \infty, \qquad \frac{q_k^{(v_m)}}{|q_k^{(v_m)}|} \longrightarrow v_m \qquad \text{and} \qquad \eta_k^{(v_m)} \longrightarrow \gamma_{p,v_m}.$$

Hence for any m there exists k_m such that $|q_{k_m}^{(v_m)}| \geq m$ while

$$\left| \frac{q_{k_m}^{(v_m)}}{|q_{k_m}^{(v_m)}|} - v_m \right| \leq \frac{1}{m} \qquad \text{and} \qquad \left| \dot{\eta}_{k_m}^{(v_m)}(0) - \dot{\gamma}_{p,v_m}(0) \right| \leq \frac{1}{m}. \tag{22}$$

Since $v_m \longrightarrow v$ we learn from (22) that $p_m := q_{k_m}^{(v_m)}$ satisfies $p_m \rightsquigarrow v$. The minimizing geodesic segment $\gamma_m := \eta_{k_m}^{(v_m)}$ connects the point p to p_m, according to (19). From (i) we thus conclude that γ_m converges to $\gamma_{p,v}([0,\infty))$ as $m \to \infty$. Consequently we obtain from (22) that

$$\gamma_{p,v_m} \xrightarrow{m\to\infty} \gamma_{p,v}.$$

Hence the map $F(v) = \dot{\gamma}_{p,v}(0)$ is continuous. Recall that the unit tangent sphere S_pM is diffeomorphic to S^{n-1}. The Brouwer degree of F as a continuous, odd map from S^{n-1} to S_pM is an odd number. In particular the degree is non-zero, and hence F is onto. □

4 Geodesics through L-points

Proposition 3.7 admits the following corollary, which completes the proof of Theorem 1.6.

Corollary 4.1. *The manifold M is diffeomorphic to $\mathbb{R}^n$. In fact, for any $p \in L$, the exponential map $\exp_p : T_pM \to M$ is a diffeomorphism and all geodesics passing through p are minimizing.*

Moreover, for any geodesic ray γ that emanates from p there is a sequence $(p_m)_{m\geq 1}$ in L with $p_m \longrightarrow \infty$ such that the geodesic segment from p to p_m tends to γ as $m \to \infty$.

Proof. According to Proposition 3.7(ii), any geodesic passing through p takes the form $\gamma_{p,v}$ for some $v \in S^{n-1}$, and hence it is a complete, minimizing geodesic. Thus the cut-locus of p is empty, and consequently $\exp_p : T_pM \to M$ is a diffeomorphism onto M. The "Moreover" part follows from Proposition 3.7(i), according to which the geodesic ray $\gamma_{p,v}([0,\infty))$ is a limit of a sequence of minimizing geodesics connecting p with points in $L \setminus \{p\}$ that tend to infinity. □

Remark 4.2. The proof of Theorem 1.6 is quite robust, and in fact the assumption that M is a Riemannian manifold in Theorem 1.6 can be weakened to the requirement that M is a Finsler manifold. Moreover, we think that for a suitable notion of a quasi-net, the space $\mathbb{R}^n$ in Theorem 1.6 may be replaced by other Hadamard manifolds. For example, while Definition 3.1 above seems specific to the Euclidean space, it actually may be replaced by the ideal boundary of a Hadamard manifold, see e.g. [2].

For $v \in S^{n-1}$ define

$$B_v(x) = \inf_{B \in \partial_v M} B(x). \tag{23}$$

Recall that a 1-Lipschitz function f foliates if M is covered by transport lines of f.

Proposition 4.3. *For any $v \in S^{n-1}$, the function B_v foliates and belongs to $\partial_v M$.*

Proof. Since $\partial_v M \neq \emptyset$ by Lemma 3.4, the infimum in (23) is well-defined, and $B_v(p) = \langle p, v \rangle$ for any $p \in L$. The function B_v is 1-Lipschitz, being the infimum of a family of 1-Lipschitz functions. Consequently,

$$B_v \in \partial_v M.$$

By Lemma 3.3 there exists a sequence $L \ni p_m \rightsquigarrow v$ narrowly, that is,

$$\langle p_m, v \rangle - |p_m| \xrightarrow{m \to \infty} 0 \qquad \text{and} \qquad |p_m| \xrightarrow{m \to \infty} \infty. \tag{24}$$

According to Lemma 3.4 we may pass to a subsequence, and assume that $d_{p_m} \longrightarrow B \in \partial_v M$. By Corollary 4.1 the cut-locus of p_m is empty for all m. Proposition 2.3(i) thus implies that B foliates. It remains to prove that $B \equiv B_v$. To this end we note that since B_v is 1-Lipschitz, for any $x \in M$ and $m \geq 1$,

$$B_v(x) \geq B_v(p_m) - d(p_m, x) = \langle p_m, v \rangle - d(p_m, x) = d_{p_m}(x) + (\langle p_m, v \rangle - |p_m|) \xrightarrow{m \to \infty} B(x),$$

where we also used (24) in the last passage. Thus $B_v \geq B$. However since $B \in \partial_v M$, the inequality $B_v \leq B$ follows from the definition (23). Hence $B \equiv B_v$. ◻

The rest of this paper is devoted almost exclusively to the proof of Theorem 1.5. From now on and until the end of Section 6 we further assume that the discrete set $L \subseteq \mathbb{R}^n$ satisfies the quasi-net condition (QN2) in addition to (QN1), and that

$$n = 2.$$

Thus M is a two-dimensional manifold homeomorphic to $\mathbb{R}^2$, by Corollary 4.1.

Lemma 4.4. *For any $p \in L$, the odd map $F(v) = \dot{\gamma}_{p,v}(0)$ from S^1 to $S_p M$ is a homeomorphism.*

Proof. The function F is continuous and onto, by Proposition 3.7(ii). We need to prove that F is one-to-one. We will use the following one-dimensional topological fact: If $h : S^1 \to S^1$ is a continuous map, and $A, B \subseteq S^1$ are two dense subsets with $A = h^{-1}(B)$ such that the restriction of h to A is one-to-one, then the function $h : S^1 \to S^1$ is one-to-one.

Let $q \in L \setminus \{p\}$ and set $v = (q-p)/|q-p| \in S^1$. We claim that for any $w \in S^1$,

$$q \in \gamma_{p,w}((0, \infty)) \qquad \Longleftrightarrow \qquad w = v. \tag{25}$$

Indeed, we know that $\gamma_{p,w}$ is a transport line of the 1-Lipschitz function $B_w \in \partial_w M$, by Proposition 3.7(iii) and Proposition 4.3. Hence, by (4) above,

$$q \in \gamma_{p,w} \qquad \Longleftrightarrow \qquad |B_w(q) - B_w(p)| = d(p,q).$$

Since $B_w(q) - B_w(p) = \langle q - p, w \rangle$ and $d(p,q) = |q - p|$, we conclude that

$$q \in \gamma_{p,w} \qquad \Longleftrightarrow \qquad w = \pm v. \tag{26}$$

By Proposition 3.7(ii), we know that $\gamma_{p,-v}((0,\infty)) = \gamma_{p,v}((-\infty,0))$. Since $B_v(q) > B_v(p)$ and since $\gamma_{p,v}$ is a transport line of B_v with $\gamma_{p,v}(0) = p$, we deduce from (26) that $q \in \gamma_{p,v}((0,\infty))$ but $q \notin \gamma_{p,v}((-\infty,0)) = \gamma_{p,-v}((0,\infty))$. Hence (25) follows from (26). Write

$$A = \left\{ \frac{q-p}{|q-p|} \, ; \, q \in L \setminus \{p\} \right\},$$

which is a dense subset of S^1 by Lemma 3.3. By (25) and by the "Moreover" part in Corollary 4.1, the set $B := F(A)$ is a dense subset of $S_p M$. For any $b \in B$, the geodesic ray emanating from p in direction $b \in S_p M$ contains a point $q \in L \setminus \{p\}$, and hence $F^{-1}(b) \subseteq S^1$ is a singleton by (25). Consequently, $A = F^{-1}(B)$ and the restriction of F to A is one-to-one. In view of the above fact, F is one-to-one. □

Let $\gamma : \mathbb{R} \to M$ be any simple curve with $\lim_{t \to \pm\infty} \gamma(t) = \infty$ (for example, any complete minimizing geodesic has this property). The curve γ induces a simple closed curve in the one-point compactification of M which is homeomorphic to the two-dimensional sphere. From the Jordan curve theorem we learn that $M \setminus \gamma$ consists of two connected components, each of which is homeomorphic to $\mathbb{R}^2$ by the Schönflies theorem.

When $\gamma_1, \gamma_2 : \mathbb{R} \to M$ are disjoint simple curves with $\lim_{t \to \pm\infty} \gamma_i(t) = \infty$ for $i = 1, 2$, the set $M \setminus (\gamma_1 \cup \gamma_2)$ thus consists of three connected components. Exactly one of these three connected components is *in the middle*, in the sense that any curve connecting the two other components, has to intersect the middle one. Denote the set of all points in this middle connected component by $\mathrm{btwn}(\gamma_1, \gamma_2)$, so as to say that a point $x \in \mathrm{btwn}(\gamma_1, \gamma_2)$ is *between* γ_1 and γ_2.

Lemma 4.5. *For any $x \in M$ and $v \in S^1$, there exist $p, q \in L$ such that $x \in \mathrm{btwn}(\gamma_{p,v}, \gamma_{q,v})$.*

Proof. By Proposition 4.3 there is a complete minimizing geodesic η with $\eta(0) = x$ which is a transport line of B_v. By the Jordan curve theorem, the curve η separates the manifold M into two connected components. The geodesic η cannot contain the entire lattice L: Otherwise, it follows from (7) that all points of L are contained in a single straight line in $\mathbb{R}^n$, and this possibility is ruled out by either (QN1) or (QN2). Hence there exists $p \in L \setminus \eta$. Write H^+ for the connected component of $M \setminus \gamma$ that contains p, and write H^- for the other connected component.

By Corollary 4.1, there exists a minimizing geodesic ray γ emanating from p that passes through x. This geodesic ray crosses the geodesic η at the point x. By Corollary 4.1, we may find a sequence $q_m \in L$ such that $q_m \longrightarrow \infty$ and such that the geodesic segment from p to q_m

tends to γ. Thus for a sufficiently large m, the geodesic segment from p to q_m crosses η at a point close to x. This minimizing geodesic segment cannot cross η twice, since η is a complete, minimizing geodesic. Hence $q := q_m \in L \cap H^-$.

We have thus found points $p \in L \cap H^+$ and $q \in L \cap H^-$. From Lemma 2.1 and Proposition 3.7(iii) we understand that the entire curve η lies between $\gamma_{p,v}$ and $\gamma_{q,v}$, and in particular $x = \eta(0) \in \mathrm{btwn}(\gamma_{p,v}, \gamma_{q,v})$. ◻

Lemma 4.6. *Let $p \in L$, $v \in S^1, 0 < \varepsilon < 1$ and write H^+ and H^- for the two connected components of $M \setminus \gamma_{p,v}$. Then there exist $R > 0$ and a unit vector $v^\perp \in S^1$ orthogonal to v such that the following holds: For any $q \in L$ with $|q| \geq R$ and for any choice of sign,*

$$\langle q/|q|, \pm v^\perp \rangle > \varepsilon \qquad \Longrightarrow \qquad q \in H^\pm. \tag{27}$$

Proof. Corollary 4.1 implies that two distinct geodesic rays emanating from p are disjoint except for their intersection at p. Therefore any geodesic ray from p is contained either in H^+ or in H^- or in $\gamma_{p,v}$. Set

$$I^\pm = \left\{ w \in S^1 \, ; \, \gamma_{p,w}((0,\infty)) \subset H^\pm \right\}. \tag{28}$$

It follows from Lemma 4.4 that $I^+ \cup I^- = S^1 \setminus \{v, -v\}$ with $I^+ \cap I^- = \emptyset$. Since $I^\pm$ is an open subset of S^1, we conclude that $I^\pm$ is an open arc in S^1 that stretches from the point v to its antipodal point $-v$. Write $v^\perp \in S^1$ for the unique vector in I^+ that is orthogonal to v.

Assume, by way of contradiction, that there exists a sequence $q_m \in L \cap (H^\mp \cup \gamma_{p,v})$ tending to infinity with

$$\langle q_m/|q_m|, \pm v^\perp \rangle > \varepsilon \qquad \text{for all } m. \tag{29}$$

Write γ_m for the minimizing geodesic from p to q_m, excluding the two endpoints. Since $q_m \in H^\mp \cup \gamma_{p,v}$ and since a geodesic ray emanating from p does not cross $\gamma_{p,v}$,

$$\gamma_m \subseteq H^\mp \cup \gamma_{p,v}. \tag{30}$$

By passing to a subsequence and using (29), we may assume that $q_m/|q_m| \longrightarrow w$ for some $w \in S^1$ with

$$\langle w, \pm v^\perp \rangle \geq \varepsilon > 0. \tag{31}$$

Proposition 3.7(i) now tells us that $\gamma_m \longrightarrow \gamma := \gamma_{p,w}((0,\infty))$, while (31) shows that $w \in I^\pm$. We thus learn from (28) that

$$\gamma \subseteq H^\pm.$$

However, from (30) we obtain $\gamma \subseteq H^\mp \cup \gamma_{p,v}$, in contradiction. ◻

Proposition 4.7. *For any $p \in L$ and $v \in S^1$,*

$$\lim_{t \to \infty} \frac{d(\gamma_{p,v}(t), L)}{t} = 0.$$

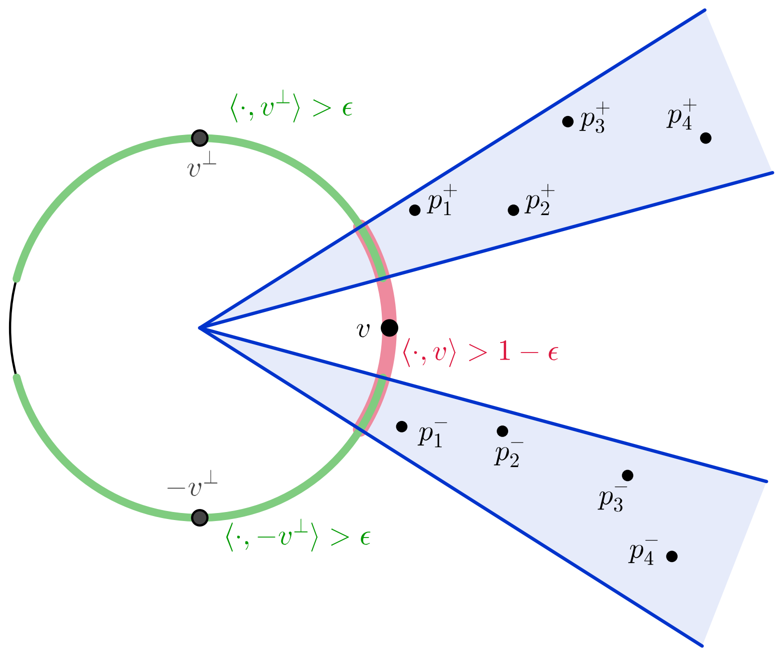

Figure 1: The L-points used in the proof of Proposition 4.7

Proof. Let $0 < \varepsilon < 1$. We will show that there exists T such that for $t > T$,

$$d(\gamma_{p,v}(t), L) \leq |p| + 21\sqrt{\varepsilon} \cdot t. \tag{32}$$

In view of Lemma 4.6, we may write H^+ and H^- for the two connected components of $M \setminus \gamma_{p,v}$, and conclude the existence of $R > 0$ and of a unit vector

$$v^\perp \in S^1$$

orthogonal to v such that (27) holds true for any $q \in L$ with $|q| \geq R$. Let us apply condition (QN2). It implies that there exist two sequences $(p_m^+)_{m\geq 1}$ and $(p_m^-)_{m\geq 1}$ in the discrete set L with

$$\left\langle \frac{p_m^\pm}{|p_m^\pm|}, \pm v^\perp \right\rangle > \varepsilon \qquad \text{and} \qquad \left\langle \frac{p_m^\pm}{|p_m^\pm|}, v \right\rangle > 1 - \varepsilon \tag{33}$$

such that $p_m^\pm \longrightarrow \infty$ and $|p_{m+1}^\pm|/|p_m^\pm| \longrightarrow 1$. From (27) and (33) we conclude that there exists $M \geq 1$ such that for all $m \geq M$,

$$p_m^\pm \in H^\pm \qquad \text{and} \qquad \frac{|p_{m+1}^\pm|}{|p_m^\pm|} < 1 + \varepsilon. \tag{34}$$

Set $T = \max_{1\leq m\leq M} \max\{|p_m^+|, |p_m^-|\}$. For $t > T$, define

$$m^\pm = m^\pm(t) := \min\{m \geq 1\, ;\, |p_m^\pm| > t\}.$$

The positive integers $m^{\pm} > M$ are well-defined as $p_m^{\pm} \longrightarrow \infty$. According to (34),

$$a := p^+_{m^+} \in H^+ \qquad \text{and} \qquad b := p^-_{m^-} \in H^-. \tag{35}$$

Since $|p^{\pm}_{m^{\pm}-1}| \leq t$, by (34) we also have

$$1 < \frac{|a|}{t} < 1 + \varepsilon \qquad \text{and} \qquad 1 < \frac{|b|}{t} < 1 + \varepsilon. \tag{36}$$

Furthermore, from (33) we know that $\langle a/|a|, v\rangle > 1 - \varepsilon$ and $\langle b/|b|, v\rangle > 1 - \varepsilon$. Hence the Euclidean distance between $a/|a|$ and $b/|b|$ is less than $4\sqrt{\varepsilon}$. From (36) we conclude that

$$d(a,b) = |a-b| \leq \left| a - \frac{|a|}{|b|} b \right| + \left| \frac{|a|}{|b|} b - b \right| \leq (1+\varepsilon)t \left| \frac{a}{|a|} - \frac{b}{|b|} \right| + (1+\varepsilon)t \left| \frac{|a|}{|b|} - 1 \right| \leq 10\sqrt{\varepsilon}t. \tag{37}$$

It follows from (35) that the minimizing geodesic between a and b intersects the curve $\gamma_{p,v}$ at a unique point $\gamma_{p,v}(t_1)$. From (37),

$$d(\gamma_{p,v}(t_1), a) \leq d(a,b) \leq 10\sqrt{\varepsilon}t. \tag{38}$$

Since $|d(p,a) - |a|| = |d(p,a) - d(a,0)| \leq d(0,p) = |p|$, by (36) we have

$$t - |p| < d(a,p) < (1+\varepsilon)t + |p|. \tag{39}$$

From (38) and (39) it follows that

$$t_1 = d(\gamma_{p,v}(t_1), p) \in (t - |p| - 10\sqrt{\varepsilon}t, (1+\varepsilon)t + |p| + 10\sqrt{\varepsilon}t).$$

Therefore,

$$|t_1 - t| \leq |p| + \varepsilon t + 10\sqrt{\varepsilon}t \leq |p| + 11\sqrt{\varepsilon}t. \tag{40}$$

Finally, by (38), (40) and the triangle inequality, since $a \in L$,

$$d(\gamma_{p,v}(t), L) \leq |t_1 - t| + d(\gamma_{p,v}(t_1), L) \leq |t_1 - t| + d(\gamma_{p,v}(t_1), a) \leq |p| + 21\sqrt{\varepsilon}t,$$

as advertised in (32). □

The large scale geometry of M is approximately Euclidean, according to the following:

Proposition 4.8. *Let $p \in L$. Then for $x, y \in \mathbb{R}^2$, writing $x = av, y = bw$ with $v, w \in S^1$ and $a, b \geq 0$, we have*

$$\lim_{r \to \infty} \frac{d(\gamma_{p,v}(ar), \gamma_{p,w}(br))}{r} = |x - y|,$$

and the convergence is locally uniform in $x, y \in \mathbb{R}^2$.

Proof. Abbreviate $\gamma_v = \gamma_{p,v}$. In order to prove the local uniform convergence, we fix no less than five sequences

$$r_m \longrightarrow \infty, \ \ S^1 \ni w_m \longrightarrow w, \ \ S^1 \ni v_m \longrightarrow v, \ \ [0,\infty) \ni a_m \longrightarrow a, \ \ [0,\infty) \ni b_m \longrightarrow b.$$

Our goal is to prove that

$$L := \lim_{m\to\infty} \frac{d(\gamma_{v_m}(a_m r_m), \gamma_{w_m}(b_m r_m))}{r_m} = |x-y|. \tag{41}$$

We may assume that the limit L on the left-hand side of (41) is attained, as an element of $\mathbb{R} \cup \{+\infty\}$, and our goal is to prove that $L = |x-y|$. Passing to a subsequence, we may assume that $\lim_m a_m r_m$ and $\lim_m b_m r_m$ exist as elements in $\mathbb{R} \cup \{+\infty\}$. First consider the case where $\lim_m a_m r_m < \infty$. In this case necessarily $a_m \longrightarrow 0 = a$, hence $x = 0$ and

$$L = \lim_{m\to\infty} \frac{d(p, \gamma_{w_m}(b_m r_m))}{r_m} = \lim_{m\to\infty} \frac{b_m r_m}{r_m} = b = |x-y|.$$

Therefore (41) holds true when $\sup_m a_m r_m < \infty$. Similarly, (41) holds true when $\sup_m b_m r_m < \infty$. We may thus assume that

$$\lim_{m\to\infty} a_m r_m = \lim_{m\to\infty} b_m r_m = +\infty. \tag{42}$$

According to Proposition 4.7, for any m we may select $p_m, q_m \in L$ with

$$\lim_{m\to\infty} \frac{d(p_m, \gamma_{v_m}(a_m r_m))}{a_m r_m} = 0 \qquad \text{and} \qquad \lim_{m\to\infty} \frac{d(q_m, \gamma_{w_m}(b_m r_m))}{b_m r_m} = 0. \tag{43}$$

By the triangle inequality, we know that

$$\left| \frac{|p_m - p|}{a_m r_m} - 1 \right| \leq \frac{d(p_m, \gamma_{v_m}(a_m r_m))}{a_m r_m} \qquad \text{and} \qquad \left| \frac{|p_m|}{a_m r_m} - \frac{|p_m - p|}{a_m r_m} \right| \leq \frac{|p|}{a_m r_m}.$$

Using (42) and (43), we have that $|p_m|/(a_m r_m) \longrightarrow 1$ and similarly $|q_m|/(b_m r_m) \longrightarrow 1$. Moreover, since $B_{v_m} \in \partial_{v_m} M$ by Proposition 4.3, and γ_{v_m} is its transport line by Proposition 3.7(iii),

$$\lim_{m\to\infty} \left\langle \frac{p_m}{|p_m|}, v_m \right\rangle = \lim_{m\to\infty} \frac{B_{v_m}(p_m)}{|p_m|} = \lim_{m\to\infty} \frac{B_{v_m}(\gamma_{v_m}(a_m r_m))}{a_m r_m} = 1. \tag{44}$$

Since $S^1 \ni v_m \longrightarrow v$, we learn from (44) that $p_m/|p_m| \longrightarrow v$. Similarly $q_m/|q_m| \longrightarrow w$. Consequently,

$$L = \lim_{m\to\infty} \frac{|p_m - q_m|}{r_m} = \lim_{m\to\infty} \left| a_m \frac{|p_m|}{a_m r_m} \frac{p_m}{|p_m|} - b_m \frac{|q_m|}{b_m r_m} \frac{q_m}{|q_m|} \right| = |av - bw| = |x-y|.$$

□

At this point in the proof we quote the area growth theorem of Bangert and Emmerich [4]. We write $D_M(x,r) = \{y \in M \, ; \, d(x,y) < r\}$, the geodesic ball of radius r centered at x. Theorem 1 from [4] reads as follows:

Theorem 4.9 (Bangert and Emmerich)**.** *Let M be a complete, two-dimensional Riemannian manifold without conjugate points, diffeomorphic to $\mathbb{R}^2$ and let $x \in M$. Then,*

$$\liminf_{r \to \infty} \frac{\text{Area}(D_M(x,r))}{\pi r^2} \geq 1,$$

with equality if and only if M is flat.

Theorem 4.9 may be viewed in the context of the isoperimetric inequality of Weil and of Beckenbach and Radó (see [24]), and it demonstrates that in two dimensions, the lack of conjugate points is sometimes a substitute for the stronger assumption of non-positive curvature.

In order to prove Theorem 1.5 it thus remains to show that M has no conjugate points, and to use the fact that the large scale geometry of M is approximately Euclidean in order to prove that $\text{Area}(D_M(x,r))/r^2 \longrightarrow \pi$ as r tends to infinity. We remark that there are surfaces whose large scale geometry is approximately Euclidean, such as $z = \sin(x)\sin(y)$ in $\mathbb{R}^3$, yet they have conjugate points and are consequently not isometric to the Euclidean plane.

5 The ideal boundary

Our goal in this section is to prove that $\partial_v M$ is a singleton for each $v \in S^1$. We begin with the following:

Lemma 5.1. *Let $v, w \in S^1, C \in \mathbb{R}$ and let $\gamma : \mathbb{R} \to M$ be a transport line of B_w (for example, when $\gamma = \gamma_{p,w}$ for some $p \in L$). Assume that for any $t > 0$,*

$$B_v(\gamma(t)) \geq t + C. \tag{45}$$

Then $v = w$.

Proof. Since γ is a transport line of B_w, by (45) there exists $C' \in \mathbb{R}$ such that

$$\min\{B_v(\gamma(t)), B_w(\gamma(t))\} \geq t + C' \qquad (t > 0). \tag{46}$$

According to Proposition 4.7, for any $t > 0$ there exists $p_t \in L$ with

$$\lim_{t \to \infty} \frac{d(\gamma(t), p_t)}{t} = 0. \tag{47}$$

Since $d(\gamma(t), \gamma(0)) = t$, we have that $|d(p_t, 0) - t| = o(t)$ as $t \to \infty$, by the triangle inequality. Moreover, from (46), (47), and the fact that $B_v \in \partial_v M$ and $B_w \in \partial_w M$ are 1-Lipschitz,

$$\begin{aligned} \min\{\langle p_t, v\rangle, \langle p_t, w\rangle\} = \min\{B_v(p_t), B_w(p_t)\} &= \min\{B_v(\gamma(t)), B_w(\gamma(t))\} + o(t) \\ &\geq t + o(t) = d(0, p_t) + o(t) = |p_t| + o(t). \end{aligned} \tag{48}$$

Since $|p_t| = t + o(t)$, we know that $|p_t|$ tends to infinity with t, and we deduce from (48) that

$$\lim_{t \to \infty} \min \left\{ \left\langle \frac{p_t}{|p_t|}, v \right\rangle, \left\langle \frac{p_t}{|p_t|}, w \right\rangle \right\} = 1. \tag{49}$$

Since $|v| = |w| = 1$, it follows from (49) that $p_t/|p_t| \longrightarrow v$ and $p_t/|p_t| \longrightarrow w$. Hence $v = w$. □

Recall that $B_{\gamma_{p,v}}$ is the Busemann function of the geodesic $\gamma_{p,v}$. The following proposition is a step in the proof that $\partial_v M$ is a singleton.

Proposition 5.2. *For any $q \in L, v \in S^1$ and $x \in M$ we have $B_{\gamma_{q,v}}(x) = B_v(x) - \langle q, v \rangle$.*

The proof of Proposition 5.2 requires several lemmas.

Lemma 5.3. *Let $f : M \to \mathbb{R}$ be a 1-Lipschitz function and let $\gamma : \mathbb{R} \to M$ be a transport line of f. Then for any $x \in M$,*

$$B_\gamma(x) + f(\gamma(0)) \le f(x).$$

Proof. Since γ is a transport line of f, we have $f(\gamma(t)) = t + f(\gamma(0))$ for all $t \in \mathbb{R}$. Since f is 1-Lipschitz, for any $x \in M$,

$$B_\gamma(x) = \lim_{t \to \infty} [t - d(x, \gamma(t))] \le \lim_{t \to \infty} [t - (f(\gamma(t)) - f(x))] = f(x) - f(\gamma(0)).$$

□

Lemma 5.4. *Let $p, q \in L$ and $v \in S^1$. Suppose that for every $t > 0$ there is a point $y_t \in L \setminus \{q\}$ with $|y_t| > t$ such that the geodesic segment from q to y_t passes through the ball $D_M(\gamma_{p,v}(t), 1/t)$. Then $y_t \rightsquigarrow v$ (i.e., $y_t/|y_t| \longrightarrow v$ as $t \to \infty$).*

Proof. Let $(t_m)_{m \ge 1}$ be an increasing sequence tending to infinity with $y_{t_m}/|y_{t_m}| \longrightarrow w \in S^1$. Our goal is to prove that $w = v$. With a slight abuse of notation we abbreviate

$$y_m = y_{t_m}.$$

Write $\eta_m : \mathbb{R} \to M$ for the minimizing geodesic with $\eta_m(0) = q$ and $\eta_m(d(q, y_m)) = y_m$, uniquely determined by Corollary 4.1. Since $y_m \rightsquigarrow w$, it follows from Proposition 3.7(i) that

$$\eta_m \xrightarrow{m \to \infty} \gamma_{q,w}. \tag{50}$$

By our assumptions, for any m there exists a point $\eta_m(s_m)$ on the geodesic segment between q and y_m with

$$\eta_m(s_m) \in D_M(\gamma_{p,v}(t_m), 1/t_m). \tag{51}$$

Since $d(\gamma_{p,v}(t_m), p) = t_m$ and $d(\eta_m(s_m), q) = s_m$, from (51) and the triangle inequality,

$$|s_m - t_m| < d(p, q) + 1/t_m. \tag{52}$$

From Proposition 3.7(iii), there exists $C \in \mathbb{R}$ such that $B_v(\gamma_{q,v}(t)) = t + C$ for all $t \in \mathbb{R}$. Since B_v is 1-Lipschitz, from (51) and (52),

$$B_v(\eta_m(s_m)) \geq B_v(\gamma_{q,v}(t_m)) - \frac{1}{t_m} = t_m + C - \frac{1}{t_m} \geq s_m + C', \tag{53}$$

with $C' = C - 2/t_1 - d(p,q)$. Since $t_m \longrightarrow \infty$, we learn from (52) that $s_m \longrightarrow \infty$ as well. Thus, for any fixed $s > 0$, there exists m with $s_m > s$ and according to (53),

$$B_v(\eta_m(s)) - s \geq B_v(\eta_m(s_m)) - s_m \geq C'. \tag{54}$$

By letting m tend to infinity we see from (50) and (54) that for all $s > 0$,

$$B_v(\gamma_{q,w}(s)) \geq s + C'.$$

Lemma 5.1 now shows that $v = w$. ◻

Lemma 5.5. *Let $q, p_1, p_2 \in L$ and $v \in S^1$. Then,*

$$\lim_{t\to\infty} [d(\gamma_{q,v}(t), p_1) - d(\gamma_{q,v}(t), p_2)] = \langle p_2 - p_1, v \rangle.$$

Proof. Abbreviate $\gamma = \gamma_{q,v}$ and let $T > 0$ be such that $\gamma(t) \neq p_i$ for $t > T$ and $i = 1, 2$. For $i = 1, 2$ and $t > T$, the geodesic ray emanating from p_i that passes through $\gamma(t)$ may be approximated arbitrarily well by a geodesic ray from p_i that passes through a faraway point of L, according to Corollary 4.1. Thus there exists

$$p_{t,i} \in L \setminus \{p_i\}$$

with $|p_{t,i}| > t$ such that the following property holds: The geodesic segment from p_i to $p_{t,i}$ passes through the ball $D_M(\gamma(t), 1/t)$. Lemma 5.4 thus implies that

$$\lim_{t\to\infty} \frac{p_{t,i}}{|p_{t,i}|} = v \qquad\qquad (i = 1, 2). \tag{55}$$

Let $z_{t,i} \in D_M(\gamma(t), 1/t)$ be a point on the geodesic segment from p_i to $p_{t,i}$, which exists by the construction of $p_{t,i}$. From the triangle inequality,

$$d(p_{t,2}, p_1) - d(p_{t,2}, p_2) = d(p_{t,2}, p_1) - d(p_{t,2}, z_{t,2}) - d(z_{t,2}, p_2) \leq d(z_{t,2}, p_1) - d(z_{t,2}, p_2), \tag{56}$$

and

$$d(p_{t,1}, p_1) - d(p_{t,1}, p_2) = d(p_{t,1}, z_{t,1}) + d(z_{t,1}, p_1) - d(p_{t,1}, p_2) \geq d(z_{t,1}, p_1) - d(z_{t,1}, p_2). \tag{57}$$

Since $z_{t,1}, z_{t,2} \in D_M(\gamma(t), 1/t)$ we obtain from (56) and (57) that for all $t > T$,

$$|p_{t,2} - p_1| - |p_{t,2} - p_2| - \frac{2}{t} \leq d(\gamma(t), p_1) - d(\gamma(t), p_2) \leq |p_{t,1} - p_1| - |p_{t,1} - p_2| + \frac{2}{t}. \tag{58}$$

However, from (55) we deduce that both the left-hand side and the right-hand side of (58) tend to $\langle p_2 - p_1, v \rangle$ as $t \to \infty$. The conclusion of the lemma thus follows from (58). ◻

Proof of Proposition 5.2. The Busemann function $B_{q,v} := B_{\gamma_{q,v}}$ is a 1-Lipschitz function. By Proposition 3.7(iii) we know that $\gamma_{q,v}$ is a transport line of B_v. Hence, by Lemma 5.3,

$$B_{q,v}(x) + \langle q, v\rangle = B_{q,v}(x) + B_v(q) \leq B_v(x) \qquad \text{for all } x \in M. \tag{59}$$

However, from Lemma 5.5, for any $p \in L$,

$$B_{q,v}(p) - B_{q,v}(q) = \lim_{t\to\infty}[d(\gamma_{q,v}(t), q) - d(\gamma_{q,v}(t), p)] = \langle p - q, v\rangle. \tag{60}$$

Since $B_{q,v}(q) = 0$, we conclude from (60) that the 1-Lipschitz function

$$x \mapsto B_{q,v}(x) + \langle q, v\rangle \qquad (x \in M)$$

belongs to $\partial_v M$. This function is bounded from above by B_v, according to (59). From the definition (23) of B_v as the smallest element in $\partial_v M$, we conclude that $B_v \equiv B_{q,v} + \langle q, v\rangle$. □

It follows from (13) and from the fact that B_v is the minimal element in $\partial_v M$ that the *maximal* element in $\partial_v M$ satisfies

$$B^v(x) := \sup_{B\in\partial_v M} B(x) = -\inf_{B\in\partial_{-v} M} B(x) = -B_{-v}(x) \qquad (x \in M). \tag{61}$$

Since $B_{-v} \in \partial_{-v} M$ by Proposition 4.3, we learn from (13) and (61) that indeed

$$B^v \in -\partial_{-v} M = \partial_v M.$$

For $v \in S^1$ we denote

$$f_v = B^v - B_v.$$

The function $f_v : M \to \mathbb{R}$ is clearly non-negative. By (61) it is also evident that $f_v = f_{-v}$.

Lemma 5.6. *Let $p \in L, v \in S^1, \varepsilon > 0$ and let $x \in M$ satisfy $f_v(x) < \varepsilon$. Then there exists $t_0 > 0$ with the following property: For any $t > t_0$ and for any point $y \in M$ lying on a minimizing geodesic segment connecting x and $\gamma_{p,v}(t)$, we have*

$$f_v(y) < \varepsilon.$$

Proof. Abbreviate $\gamma = \gamma_{p,v}$, and for $s > 0$ and $x \in M$ denote

$$\delta(s, x) = d(\gamma(s), x) + d(\gamma(-s), x) - 2s.$$

The function $\delta(s, x)$ is non-increasing in s. Moreover, since $\gamma_{p,-v}(s) = \gamma_{p,v}(-s)$ by Proposition 3.7(ii), we deduce that for any $x \in M$,

$$\lim_{s\to\infty} \delta(s, x) = -B_{\gamma_{p,v}}(x) - B_{\gamma_{p,-v}}(x) = -B_v(x) - B_{-v}(x) = B^v(x) - B_v(x) = f_v(x),$$

where we used (61) and Proposition 5.2 in the last passages. Since $\delta(t, x) \searrow f_v(x) < \varepsilon$ as $t \to \infty$, there exists t_0 such that for any $t > t_0$,

$$\delta(t, x) < \varepsilon. \tag{62}$$

Fix $t > t_0$. Since the point y lies on a minimizing geodesic connecting x and $\gamma(t)$, it follows from (62) and the triangle inequality that

$$\begin{aligned}\delta(t, y) = d(\gamma(t), y) + d(\gamma(-t), y) - 2t &= d(\gamma(t), x) - d(x, y) + d(\gamma(-t), y) - 2t \\ &\leq d(\gamma(t), x) + d(\gamma(-t), x) - 2t = \delta(t, x) < \varepsilon.\end{aligned}$$

However, $\delta(s, y)$ is non-increasing in s. Therefore,

$$f_v(y) = \lim_{s\to\infty} \delta(s, y) \leq \delta(t, y) < \varepsilon.$$

□

In the proof of the following proposition we rely heavily on the fact that any geodesic through a point in L is minimizing, according to Corollary 4.1.

Proposition 5.7. *For any $v \in S^1$ we have $B^v \equiv B_v$ and hence $\partial_v M$ is a singleton.*

Proof. Let $x \in M$ and $\varepsilon > 0$. Our goal is to prove that $f_v(x) = B^v(x) - B_v(x) < \varepsilon$. To this end we use Lemma 4.5, according to which there exist $p, q \in L$ such that

$$x \in S := \text{btwn}(\gamma_{p,v}, \gamma_{q,v}). \tag{63}$$

We claim that:

(64) The geodesic from q to any point in $\gamma_{p,v}$ is pointing towards S at the point q.

(65) The geodesic segment from q to x is pointing towards S at the point q, and it does not intersect $\gamma_{p,v}$.

Indeed, by Corollary 4.1, any complete geodesic through q which is not $\gamma_{q,v}$ cannot intersect $\gamma_{q,v} \setminus \{q\}$, and (64) follows. We learn from (63) that the geodesic from q to x cannot intersect $\gamma_{q,v} \setminus \{q\}$, hence it is pointing towards S at the point q. Moreover, this geodesic cannot cross $\gamma_{p,v}$ twice, and it ends at a point in S, and therefore this geodesic segment cannot intersect $\gamma_{p,v}$ at all. This proves (65).

Proposition 5.2 tells us that $B_v = B_{\gamma_{p,v}} + \langle p, v\rangle$ and that $B_{-v} = B_{\gamma_{p,-v}} - \langle p, v\rangle$. Since $\gamma_{p,-v}(t) = \gamma_{p,v}(-t)$, this means that for any $y \in M$,

$$B_v(y) = \lim_{t\to\infty} [C_t - d(y, \gamma_{p,v}(t))] \qquad \text{and} \qquad B_{-v}(y) = \lim_{t\to\infty} [C'_t - d(y, \gamma_{p,v}(-t))] \tag{66}$$

for $C_t = t + \langle p, v\rangle$ and $C'_t = t - \langle p, v\rangle$. Recall that B_v and B_{-v} foliate by Proposition 4.3, and that $\nabla B_v(q) = \dot{\gamma}_{q,v}(0) = -\dot{\gamma}_{q,-v}(0) = -\nabla B_{-v}(q)$ thanks to Proposition 3.7. In view of (66) we

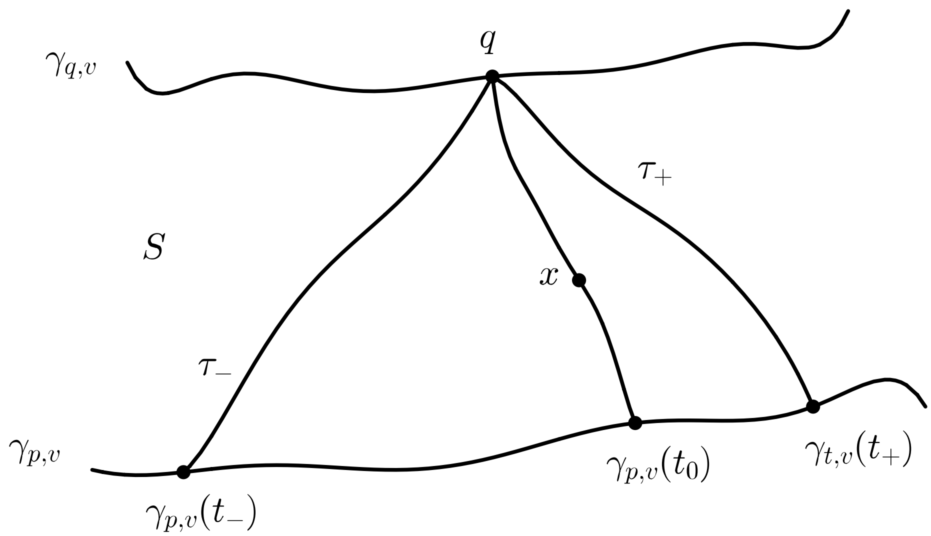

Figure 2: The geometric construction from the proof of Proposition 5.7

may invoke Proposition 2.3(ii) and conclude the following: The minimizing geodesic from q to $\gamma_{p,v}(t)$ tends to $\gamma_{q,v}$ as $t \to \infty$, and the minimizing geodesic from q to $\gamma_{p,v}(-t)$ tends to $\gamma_{q,-v}$ as $t \to \infty$.

In other words, the angle with $\gamma_{q,v}$ of the geodesic from q to $\gamma_{p,v}(t)$ tends to zero as $t \to \infty$, and the angle with $\gamma_{q,v}$ of the geodesic from q to $\gamma_{p,v}(-t)$ tends to π as $t \to \infty$.

Claim: There exists $t_0 \in \mathbb{R}$ such that x lies on the geodesic segment from q to $\gamma_{p,v}(t_0)$.

Indeed, by continuity of the angle, for any given angle $\alpha \in (0, \pi)$ there exists $t \in \mathbb{R}$ such that the angle with $\gamma_{q,v}$ of the geodesic from q to $\gamma_{p,v}(t)$ equals α. This geodesic from q to $\gamma_{p,v}(t)$ is pointing towards S at the point q, by (64). We conclude that for any unit vector $u \in S_qM$ that is pointing towards S, there exists $y \in \gamma_{p,v}$ such that the geodesic from q to y is tangent to $u \in S_qM$.

We know from (65) that the geodesic from q to x is pointing towards S at the point q. We thus learn from the previous paragraph that there exists $y \in \gamma_{p,v}$ such that the geodesic segment from q to x coincides near q with the geodesic segment from q to y. By writing $y = \gamma_{p,v}(t_0)$, the claim follows from (65).

Since $B_v, B^v \in \partial_v M$ we know that $B^v(q) = B_v(q)$ and hence $f_v(q) = 0 < \varepsilon$. Lemma 5.6 states that there exists $t_+ > t_0$ and a geodesic segment τ_+ that connects the point q with the point $\gamma_{p,v}(t_+)$, such that

$$f_v(y) < \varepsilon \qquad \text{for all } y \in \tau_+. \tag{67}$$

Similarly, by Lemma 5.6 there exists $t_- < t_0$ and a geodesic segment τ_- that connects q and $\gamma_{p,v}(t_-) = \gamma_{p,-v}(-t_-)$ such that

$$f_v(y) = f_{-v}(y) < \varepsilon \qquad \text{for all } y \in \tau_-. \tag{68}$$

To avoid ambiguity, we stipulate that the geodesic segment $\tau_\pm$ contains its endpoints q and $\gamma_{p,v}(t_\pm)$. We form a geodesic triangle $T \subset M$, a bounded open set whose boundary ∂T consists of the three edges τ_-, τ_+ and $\gamma_{p,v}([t_-, t_+])$. The point q is a vertex of this triangle. It follows from Corollary 4.1 that any interior point of the geodesic segment from q to a point in the edge $\gamma_{p,v}((t_-, t_+))$, belongs to T.

Since $t_- < t_0 < t_+$, the point $\gamma_{p,v}(t_0)$ is located between the points $\gamma_{p,v}(t_-)$ and $\gamma_{p,v}(t_+)$ along the curve $\gamma_{p,v}$. Hence $\gamma_{p,v}(t_0)$ is an interior point of the edge of the triangle T that is opposite the vertex q. Since x is an interior point of the geodesic from q to $\gamma_{p,v}(t_0)$, we conclude that $x \in T$. By (67) and (68), we know that

$$f_v(y) < \varepsilon \qquad\qquad \text{for all } y \in \tau_- \cup \tau_+. \tag{69}$$

Since $B^v = -B_{-v}$ foliates, there exists a transport line η of B^v with $\eta(0) = x \in T$. Since $\eta(t)$ tends to infinity as $t \to \infty$, there exists $t > 0$ such that

$$\eta(t) \in \partial T. \tag{70}$$

However, $\gamma_{p,v}$ is a transport line of $B^v \in \partial_v M$, by Proposition 3.7(iii), and hence η and $\gamma_{p,v}$ are disjoint transport lines of B^v, by Lemma 2.1. We thus conclude from (70) that

$$\eta(t) \in \partial T \setminus \gamma_{p,v} \subseteq \tau_- \cup \tau_+. \tag{71}$$

However, since B_v is 1-Lipschitz, by (69) and (71),

$$f_v(x) = B^v(\eta(0)) - B_v(\eta(0)) = B^v(\eta(t)) - t - B_v(\eta(0)) \leq B^v(\eta(t)) - B_v(\eta(t)) = f_v(\eta(t)) < \varepsilon.$$

This completes the proof that $B^v \equiv B_v$ in M. Since B^v is the maximal element of $\partial_v M$ while B_v is the minimal element, $\partial_v M$ is a singelton. ☐

6 No conjugate points

Proposition 5.7 will be used in order to show that M has no conjugate points. First we need:

Lemma 6.1. *For any $x \in M$, the map $v \mapsto B_v(x)$ is continuous in $v \in S^1$, as well as the map $S^1 \ni v \mapsto \nabla B_v(x) \in S_x M$. The latter map is in fact a homeomorphism.*

Proof. Fix $x \in M$. Recall that the 1-Lipschitz function B_v vanishes at 0 for any $v \in S^1$. Let $v_m \longrightarrow v$ be a sequence in S^1 such that $(B_{v_m}(x))_{m \geq 1}$ is a convergent sequence. By the Arzela-Ascoli theorem, we may pass to a subsequence and assume that $(B_{v_m})_{m\geq 1}$ converges locally-uniformly to some limit function B, and our goal is to prove that $B \equiv B_v$. In view of Proposition 5.7, it suffices to prove that $B \in \partial_v M$. The function B is 1-Lipschitz, since it is a pointwise limit of 1-Lipschitz functions. For any $p \in L$ we have

$$B(p) = \lim_{m\to\infty} B_{v_m}(p) = \lim_{m\to\infty} \langle p, v_m \rangle = \langle p, v \rangle,$$

and hence $B \in \partial_v M$. We have thus proved that the map $v \mapsto B_v(x)$ is continuous in $v \in S^1$, for any fixed $x \in M$. Lemma 2.4 now implies the continuity of the map

$$S^1 \ni v \mapsto \nabla B_v(x) \in S_x M.$$

The map $v \mapsto \nabla B_v(x)$ is one-to-one, because if $\nabla B_v(x) = \nabla B_w(x)$, then the transport line of B_v through x coincides with that of B_w, and hence $v = w$ in view of Lemma 5.1. From (13) we know that $-B_v \in \partial_{-v} M$, hence $-B_v = B_{-v}$ by Proposition 5.7. Therefore $v \mapsto \nabla B_v(x)$ is a continuous, odd, one-to-one map from S^1 to $S_x M$, which is consequently a homeomorphism. □

Corollary 6.2. *All geodesics in M are minimizing, so there are no conjugate points in M.*

Proof. Let $x \in M$ and let γ be a complete geodesic with $\gamma(0) = x$. By Lemma 6.1 there exists $v \in S^1$ such that $\dot{\gamma}(0) = \nabla B_v(x)$. By Proposition 4.3, the geodesic γ is a transport line of B_v, and hence it is a minimizing geodesic. Since a complete, minimizing geodesic cannot contain a pair of conjugate points, there are no conjugate points in M. □

Given $x \in \mathbb{R}^2$ and $r > 0$, we write $x = av$ for $a \geq 0$ and $v \in S^1$ and define

$$T_r(x) = \gamma_{0,v}(ar).$$

Then $T_r : \mathbb{R}^2 \to M$ is a homeomorphism, by Corollary 4.1 and Lemma 4.4. Moreover, if $D \subseteq \mathbb{R}^2$ is the open Euclidean unit disc centered at the origin, and $D_M(0, r) \subseteq M$ is the open geodesic ball of radius r centered at 0 in M, then $T_r(D) = D_M(0, r)$. For $x, y \in \mathbb{R}^2$ and for $r > 0$ denote

$$d_r(x, y) = \frac{d(T_r(x), T_r(y))}{r}.$$

Recall that we already discussed the large-scale geometry of M. In fact, Proposition 4.8 directly implies the following:

Corollary 6.3. *For any $x, y \in \mathbb{R}^2$,*

$$\lim_{r \to \infty} d_r(x, y) = |x - y|$$

and the convergence is locally uniform in $x, y \in \mathbb{R}^2$.

Write M_r for the Riemannian manifold obtained from M by multiplying the metric tensor by a factor of $1/r^2$. Then $(\mathbb{R}^2, d_r)$ is a metric space isometric to M_r via the map T_r, where M_r is a complete, connected Riemannian surface in which all geodesics are minimizing. Write μ_r for the area measure on $\mathbb{R}^2$ corresponding to the this isometry. That is, $r^2 \cdot \mu_r$ is the measure on $\mathbb{R}^2$ obtained by pulling back the Riemannian area measure on M under the homeomorphism T_r.

Proof of Theorem 1.5. Recall that $D \subseteq \mathbb{R}^2$ is the open Euclidean unit disc centered at the origin. We claim that

$$\lim_{r\to\infty} \mu_r(D) = \pi. \tag{72}$$

Indeed, from Corollary 6.3 we know that d_r tends locally uniformly to the Euclidean metric in $\mathbb{R}^2$ as $r \to \infty$. Moreover, for any r, the topology induced by d_r on $\mathbb{R}^2$ is the standard one, and the metric space $(\mathbb{R}^2, d_r)$ is isometric to a complete, connected, 2-dimensional Riemannian manifold in which all geodesics are minimizing. Proposition 7.1, stated and proven in the appendix below, thus yields (72). Consequently, by the definition of T_r, d_r and μ_r,

$$\frac{\text{Area}(D_M(0,r))}{r^2} = \mu_r(D) \xrightarrow{r\to\infty} \pi. \tag{73}$$

Corollary 4.1 states that M is diffeomorphic to $\mathbb{R}^2$. According to Corollary 6.2 there are no conjugate points in M. In view of (73) we may apply Theorem 4.9, due to Bangert and Emmerich, and conclude that M is flat. The only flat surface in which all geodesics are minimizing is the Euclidean plane $\mathbb{R}^2$. □

Proof of Corollary 1.2. Suppose by contradiction that X embeds isometrically in a complete, 2-dimensional, Riemannian manifold $\tilde{M}$. Since all distances in X are finite, we may assume that $\tilde{M}$ is connected. Since X contains a net in a two-dimensional affine plane, $\tilde{M}$ is necessarily isometric to the Euclidean plane $\mathbb{R}^2$ by Theorem 1.5. Hence for any four points $x_1, x_2, x_3, x_4 \in \tilde{M}$, abbreviating $d_{ij} = d(x_i, x_j)$, the 4×4 matrix

$$\left(\frac{-d_{ij}^2 + d_{i4}^2 + d_{j4}^2}{2} \right)_{i,j=1,\dots,4} \tag{74}$$

is of rank at most two. Indeed, the matrix in (74) is the Gram matrix of four points in a Euclidean plane. However, since $X \subseteq \mathbb{R}^3$ is not contained in a two-dimensional affine plane, there exist four points $x_1, x_2, x_3, x_4 \in X$ whose affine span in $\mathbb{R}^3$ is three-dimensional. For these four points, the matrix in (74) has rank 3, in contradiction. □

7 Appendix: Continuity of area

Suppose that for any $m \geq 1$ we are given a metric d_m on $\mathbb{R}^2$, such that the following hold:

(i) For any m, the metric d_m induces the standard topology on $\mathbb{R}^2$.

(ii) For any $x, y \in \mathbb{R}^2$ we have $d_m(x,y) \longrightarrow |x-y|$ as $m \to \infty$, and the convergence is locally uniform.

(iii) For any m, the metric space $(\mathbb{R}^2, d_m)$ is isometric to a complete, connected, 2-dimensional Riemannian manifold in which all geodesics are minimizing.

Write Area_m for the Riemannian area measure on $\mathbb{R}^2$ that corresponds to d_m under the above isometry.

Proposition 7.1. *For* $D = D(0,1) = \{x \in \mathbb{R}^2 \, ; \, |x| < 1\}$ *we have* $\text{Area}_m(D) \longrightarrow \pi$ *as* $m \to \infty$.

We were unable to find a proof of Proposition 7.1 in the literature, even though Ivanov's paper [22] contains a deeper result which "almost" implies this proposition. A proof of Proposition 7.1 is thus provided in this Appendix.

By a d_m-geodesic in $\mathbb{R}^2$ we mean a geodesic with respect to the metric d_m. The d_m-length of a d_m-rectifiable curve γ is denoted by $\text{Length}_m(\gamma)$. A set $K \subseteq \mathbb{R}^2$ is d_m-convex if the intersection of any d_m-geodesic with K is connected. All d_m-geodesics are minimizing, and each complete d_m-geodesic divides $\mathbb{R}^2$ into two connected components. Each of these connected components is a d_m-convex, open set called a d_m-half-plane. The intersection of finitely many d_m-half-planes, if bounded and non-empty, is called a d_m-polygon. Note that our polygons are always open and convex. The boundary of any d_m-polygon consists of finitely many vertices and the same number of edges, and each edge is a d_m-geodesic segment.

Write $\mathcal{G}_m$ for the collection of all complete d_m-geodesics in $\mathbb{R}^2$, where we identify between two geodesics if they differ by an orientation-preserving reparametrization. Write σ_m for the étendue measure on $\mathcal{G}_m$, see Kleinbock and Kuperberg [24, Section 5.2] and Álvarez-Paiva and Berck [1, Section 5]) and references therein for the basic properties of this measure, and for the formulae of Santaló and Crofton. The Santaló formula implies that for any open set $U \subseteq \mathbb{R}^2$,

$$\text{Area}_m(U) = \frac{1}{2\pi} \int_{\mathcal{G}_m} \text{Length}_m(\gamma \cap U) d\sigma_m(\gamma).$$

The Crofton formula implies that for any d_m-polygon $P \subseteq \mathbb{R}^2$,

$$\text{Perimeter}_m(P) = \frac{1}{2} \cdot \sigma_m \left(\{ \gamma \in \mathcal{G}_m \, ; \, \gamma \cap P \neq \emptyset \} \right),$$

where $\text{Perimeter}_m(P) = \text{Length}_m(\partial P)$. When we discuss Area, Length, Perimeter or polygons without the subscript m we refer to the usual Euclidean geometry in $\mathbb{R}^2$. Write $\mathcal{G}$ for the collection of all lines in $\mathbb{R}^2$, where we identify between two lines if they differ by an orientation-preserving reparametrization. Write σ for the Euclidean étendue measure on $\mathcal{G}$. We require the following Euclidean lemma:

Lemma 7.2. *Let* $\tilde{\sigma}$ *be a Borel measure on* $\mathcal{G}$ *such that for any convex polygon* $P \subseteq D(0,2) \subseteq \mathbb{R}^2$,

$$\text{Perimeter}(P) = \frac{1}{2} \cdot \tilde{\sigma} \left(\{ \ell \in \mathcal{G} \, ; \, \ell \cap P \neq \emptyset \} \right). \tag{75}$$

Then,

$$\frac{1}{2\pi} \int_{\mathcal{G}} \text{Length}(\ell \cap D) d\tilde{\sigma}(\ell) = \text{Area}(D) = \pi. \tag{76}$$

Proof. For any linear, orthogonal transformation $U : \mathbb{R}^2 \to \mathbb{R}^2$, the perimeter of the polygon $U(P) \subset \mathbb{R}^2$ is the same as the perimeter of P. Hence formula (75) holds true with $\tilde{\sigma}$ replaced by $U_*\tilde{\sigma}$, where by $U_*\tilde{\sigma}$ we mean the push-forward of $\tilde{\sigma}$ under the map U acting on $\mathcal{G}$ by rotating lines. Moreover, since $U(D) = D$, replacing $\tilde{\sigma}$ by $U_*\tilde{\sigma}$ does not change the value of the integral on the left-hand side of (76).

We may thus replace the measure $\tilde{\sigma}$ by the average of $U_*\tilde{\sigma}$ over $U \in O(2)$, and assume from now on that $\tilde{\sigma}$ is a rotationally-invariant measure on $\mathcal{G}$. The validity of (75) for any convex polygon $P \subset D(0,2)$ implies its validity for all convex sets in the disc $D(0,2)$. Indeed, both the left-hand side and the right-hand side of (75) are monotone in the convex set P under inclusion, and convex polygons are dense in the class of all convex subsets of $D(0,2)$. Consequently, for any $0 < \rho < 2$,

$$2\pi\rho = \text{Perimeter}(D(0,\rho)) = \frac{1}{2} \cdot \tilde{\sigma}\left(\{\ell \in \mathcal{G}\,;\, \ell \cap D(0,\rho) \neq \emptyset\}\right). \tag{77}$$

For $\ell \in \mathcal{G}$ write $r(\ell) = \inf_{x \in \ell} |x| \in [0,\infty)$. We may reformulate (77) as follows:

$$\tilde{\sigma}\left(\{\ell \in \mathcal{G}\,;\, r(\ell) < \rho\}\right) = \sigma\left(\{\ell \in \mathcal{G}\,;\, r(\ell) < \rho\}\right) \qquad \text{for any } 0 < \rho < 2. \tag{78}$$

Since both σ and $\tilde{\sigma}$ are rotationally-invariant measures on $\mathcal{G}$, they are completely determined by their push-forward under the map $\ell \mapsto r(\ell)$. From (78) we learn that $\tilde{\sigma}$ coincides with σ on the set $\mathcal{G} \cap r^{-1}([0,2))$. By the Satanló formula for σ,

$$\frac{1}{2\pi}\int_{\mathcal{G}} \text{Length}(\ell \cap D)\,d\tilde{\sigma}(\ell) = \frac{1}{2\pi}\int_{\mathcal{G}} \text{Length}(\ell \cap D)\,d\sigma(\ell) = \text{Area}(D) = \pi.$$

□

Write $(x,y) \subseteq \mathbb{R}^2$ for the Euclidean interval between $x, y \in \mathbb{R}^2$ excluding the endpoints, and $[x,y] = (x,y) \cup \{x,y\}$. We similarly write $[x,y]_m$ and $(x,y)_m$ for the d_m-geodesic between x and y, with and without the endpoints. We claim that for any $x, y \in \mathbb{R}^2$,

$$[x,y]_m \xrightarrow{m \to \infty} [x,y] \tag{79}$$

in the Euclidean Hausdorff metric (the Hausdorff metric is defined, e.g., in [6]). Indeed, for any $0 \leq \lambda \leq 1$, the point on $[x,y]_m$ whose d_m-distance from x equals $\lambda \cdot d_m(x,y)$ must converge to the point on $[x,y]$ whose Euclidean distance from x equals $\lambda \cdot d(x,y)$. It follows from our assumptions that the convergence is uniform in λ, and (79) follows. Moreover, it follows that the Hausdorff convergence in (79) is locally uniform in $x, y \in \mathbb{R}^2$.

Write $\overline{A}$ for the closure of a set A. The Euclidean ε-neighborhood of a subset $A \subseteq \mathbb{R}^2$ is the collection of all $x \in \mathbb{R}^2$ with $d(x,A) < \varepsilon$ where $d(x,A) = \inf_{y \in A} |x - y|$. Given a convex polygon $P \subseteq \mathbb{R}^2$, for a sufficiently large m we define $P^{(m)} \subseteq \mathbb{R}^2$ to be the d_m-polygon with the same vertices as P. We need m to be sufficiently large in order to guarantee that no vertex of P is in the d_m-convex hull of the other vertices.

Lemma 7.3. *Let $P_0, P_1 \subseteq D(0,3)$ be convex polygons such that $\overline{P_0} \subseteq P_1$. Then there exist $m_0 \geq 1$ and $\varepsilon > 0$ such that the following holds: For any $m \geq m_0$ and any $x, x', y, y' \in D(0,3)$ with $|x - x'| < \varepsilon$ and $|y - y'| < \varepsilon$,*

$$(x, y) \cap P_0 \neq \emptyset \qquad \Longrightarrow \qquad (x', y')_m \cap P_1^{(m)} \neq \emptyset,$$

and

$$(x', y')_m \cap P_0^{(m)} \neq \emptyset \qquad \Longrightarrow \qquad (x, y) \cap P_1 \neq \emptyset.$$

Proof. From the Hausdorff convergence in (79) it follows that for a sufficiently large m, the closure of $P_0^{(m)} \cup P_0$ is contained in $P_1 \cap P_1^{(m)}$. In fact, there exist $\delta > 0$ and $m_1 \geq 1$ such that for $m \geq m_1$, the Euclidean δ-neighborhood of $P_0^{(m)} \cup P_0$ is contained in $P_1 \cap P_1^{(m)}$.

Set $\varepsilon = \delta/2$. Since the convergence in (79) is uniform in $x, y \in D(0,3)$, there exists $m_0 \geq m_1$ such that for any $m \geq m_0$ and $x', y' \in D(0,3)$, the Hausdorff distance between (x', y') and $(x', y')_m$ is at most ε. Thus for any $m \geq m_0$ and $x, x', y, y' \in D(0,3)$ with $|x - x'| < \varepsilon$ and $|y - y'| < \varepsilon$, the Hausdorff distance between (x, y) and (x', y') is at most ε, and by the triangle inequality, the Hausdorff distance between (x, y) and $(x', y')_m$ is at most $2\epsilon = \delta$.

Hence if (x, y) intersects P_0, then $(x', y')_m$ intersects the Euclidean δ-neighborhood of P_0, which is contained in $P_1^{(m)}$. Similarly, if $(x', y')_m$ intersects $P_0^{(m)}$, then (x, y) intersects the δ-neighborhood of $P_0^{(m)}$ which is contained in P_1. ☐

Lemma 7.4. *Let $K \subseteq \mathbb{R}^2$ be a bounded, open, convex set. Then there exist d_m-polygons $K_m^{\pm} \subseteq \mathbb{R}^2$ for $m \geq 1$, real numbers $\varepsilon_m \searrow 0$ and $m_0 \geq 1$, such that for any $m \geq m_0$ the following hold:*

$$K_m^- \subseteq K \subseteq K_m^+,$$

both boundaries $\partial K_m^{\pm}$ are ε_m-close to ∂K in the Euclidean Hausdorff metric, and the d_m-perimeters of $K_m^{\pm}$ differ from Perimeter(K) *by at most ε_m.*

Proof. It suffices to show that for any fixed $\varepsilon > 0$ there exist $m_0 \geq 1$ and d_m-polygons $K_m^{\pm} \subset \mathbb{R}^2$ such that for any $m \geq m_0$,

$$K_m^- \subseteq K \subseteq K_m^+,$$

and both boundaries $\partial K_m^{\pm}$ are ε-close to ∂K in the Euclidean Hausdorff metric, and the d_m-perimeters of $K_m^{\pm}$ differ from Perimeter(K) by at most ε.

Fix $\varepsilon > 0$. We may pick finitely many points in ∂K, cyclically ordered, such that when connecting each point via a segment to its nearby points, the result is a convex polygon whose boundary is $(\varepsilon/2)$-close to ∂K in the Hausdorff metric. We may also require that the perimeter of this convex polygon differs from Perimeter(K) by at most $\varepsilon/2$.

We slightly move these finitely many points inside K, and replace the segments between the points by d_m-geodesics. For a sufficiently large m, this defines a d_m-polygon K_m^-. It follows from (79) that for a sufficiently large m, the boundary ∂K_m^- is ε-close to ∂K in the Euclidean Hausdorff metric, the d_m-perimeter of K_m^- differs from Perimeter(K) by at most ε, and $K_m^- \subseteq K$.

We still need to construct K_m^+. Approximate K by a convex polygon containing the closure of K in its interior, whose boundary is $(\varepsilon/2)$-close to ∂K in the Hausdorff metric, and whose perimeter differs from Perimeter(K) by at most $\varepsilon/2$. Replace the edges of this polygon by d_m-geodesics in order to form K_m^+. It follows from (79) that for a sufficiently large m, the d_m-convex set K_m^+ has the desired properties. □

We apply Lemma 7.4 for the unit disc $D \subseteq \mathbb{R}^2$, and obtain two d_m-polygons $D_m^\pm$ with $D_m^- \subseteq D \subseteq D_m^+$ that satisfy the conclusions of the lemma. It follows from (79) that

$$\text{Length}_m((x,y)_m \cap D_m^\pm) \xrightarrow{m\to\infty} \text{Length}((x,y)\cap D)), \tag{80}$$

and the convergence is locally uniform in $x, y \in \mathbb{R}^2$. Let us fix a convex polygon $K \subseteq \mathbb{R}^2$ such that $\overline{D(0,2)} \subseteq K$ and $\overline{K} \subseteq D(0,3)$. We apply Lemma 7.4 and obtain d_m-polygons $K_m = K_m^+$ for $m \geq 1$ that approximate K. For a set $A \subseteq \mathbb{R}^2$ denote

$$\mathcal{G}(A) = \{\ell \in \mathcal{G}\,;\, \ell \cap A \neq \emptyset\} \qquad \text{and} \qquad \mathcal{G}_m(A) = \{\gamma \in \mathcal{G}_m\,;\, \gamma \cap A \neq \emptyset\}.$$

Definition 7.5. *Define the map* $T_K : \mathcal{G}(K) \to \partial K \times \partial K \subset \mathbb{R}^2 \times \mathbb{R}^2$ *by*

$$T_K(\ell) = (a(\ell), b(\ell))$$

where $\ell \cap \partial K = \{a(\ell), b(\ell)\}$ *and the line* ℓ *is oriented from the point* $a(\ell)$ *towards the point* $b(\ell)$. *We analogously define the map* $T_m : \mathcal{G}_m(K_m) \to \partial K_m \times \partial K_m \subset \mathbb{R}^2 \times \mathbb{R}^2$ *via*

$$T_m(\gamma) = (a(\gamma), b(\gamma))$$

where $\gamma \cap \partial K_m = \{a(\gamma), b(\gamma)\}$ *and the geodesic* γ *is oriented from* $a(\gamma)$ *towards* $b(\gamma)$.

Denote by μ the push-forward of $\sigma|_{\mathcal{G}(K)}$ under the map T_K, and denote by μ_m the push-forward of $\sigma_m|_{\mathcal{G}(K_m)}$ under the map T_m. By Lemma 7.4 and the Crofton formula,

$$\frac{1}{2}\cdot \mu_m(\mathbb{R}^2 \times \mathbb{R}^2) = \text{Perimeter}_m(K_m) \xrightarrow{m\to\infty} \text{Perimeter}(K) = \frac{1}{2}\cdot \mu(\mathbb{R}^2 \times \mathbb{R}^2). \tag{81}$$

For a convex polygon $P \subseteq \mathbb{R}^2$ we write $\mathcal{F}(P) \subseteq \partial K \times \partial K$ for the collection of all pairs of points $x \neq y \in \partial K$ with $(x,y) \cap P \neq \emptyset$. For a d_m-polygon P we denote by $\mathcal{F}_m(P) \subseteq \partial K_m \times \partial K_m$ the collection of all pairs of points $x \neq y \in \partial K_m$ with $(x,y)_m \cap P \neq \emptyset$. Note that if $\overline{P} \subseteq K_m$ then by the Crofton formula,

$$\frac{1}{2}\cdot \mu_m(\mathcal{F}_m(P)) = \text{Perimeter}_m(P). \tag{82}$$

For a subset $A \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ and $\varepsilon > 0$ we write $\mathcal{N}_\varepsilon(A) \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ for the Euclidean ε-neighborhood, i.e., the collection of all $(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2$ for which there exists $(z,w) \in A$ with $|x - z| < \varepsilon$ and $|y - w| < \varepsilon$.

Lemma 7.6. *Fix two convex polygons $P_0, P_1 \subseteq \mathbb{R}^2$ with $\overline{P_0} \subseteq P_1$ and $\overline{P_1} \subseteq K$. For $i = 0, 1$ abbreviate $\mathcal{F}_i = \mathcal{F}(P_i)$. Then there exists $\varepsilon_0 > 0$ such that*

$$\limsup_{m\to\infty} \mu_m(\mathcal{N}_{\varepsilon_0}(\mathcal{F}_0)) \leq \mu(\mathcal{F}_1) = 2 \cdot \text{Perimeter}(P_1). \tag{83}$$

Furthermore, for any $0 < \varepsilon < \varepsilon_0$,

$$\liminf_{m\to\infty} \mu_m(\mathcal{N}_{\varepsilon}(\mathcal{F}_1)) \geq \mu(\mathcal{F}_0) = 2 \cdot \text{Perimeter}(P_0). \tag{84}$$

Proof. Recall that $P^{(m)} \subseteq \mathbb{R}^2$ was defined to be the d_m-polygon with the same vertices as P, which is well-defined for a sufficiently large m. Write $\mathcal{F}_i^{(m)} = \mathcal{F}_m(P_i^{(m)})$. According to Lemma 7.3 there exist $\varepsilon_0 > 0$ and $m_0 \geq 1$ such that for any $m \geq m_0$,

$$\mathcal{N}_{\varepsilon_0}(\mathcal{F}_0) \cap (\partial K_m \times \partial K_m) \subseteq \mathcal{F}_1^{(m)} \qquad \text{and} \qquad \mathcal{N}_{\varepsilon_0}(\mathcal{F}_0^{(m)}) \cap (\partial K \times \partial K) \subseteq \mathcal{F}_1. \tag{85}$$

By increasing m_0 if necessary, we may assume that $\overline{P_i^{(m)}} \subseteq K \subseteq K_m$ for all $m \geq m_0$ and $i = 0, 1$. Using (85) and (82),

$$\mu_m(\mathcal{N}_{\varepsilon_0}(\mathcal{F}_0)) \leq \mu_m(\mathcal{F}_1^{(m)}) = 2 \cdot \text{Perimeter}_m(P_1^{(m)}) \xrightarrow{m\to\infty} 2 \cdot \text{Perimeter}(P_1).$$

By Lemma 7.4, for any $0 < \varepsilon < \varepsilon_0$ there exists $m_1 \geq m_0$ such that for any $m \geq m_1$, the Euclidean Hausdorff distance between ∂K_m and ∂K is at most ε. From (85) we obtain that

$$\mathcal{N}_{\varepsilon}(\mathcal{F}_1) \cap (\partial K_m \times \partial K_m) \supseteq \mathcal{F}_0^{(m)}.$$

Hence,

$$\mu_m(\mathcal{N}_{\varepsilon}(\mathcal{F}_1)) \geq \mu_m(\mathcal{F}_0^{(m)}) = 2 \cdot \text{Perimeter}_m(P_0^{(m)}) \xrightarrow{m\to\infty} 2 \cdot \text{Perimeter}(P_0).$$

This completes the proof of (83) and (84). □

Proof of Proposition 7.1. By passing to a subsequence, we may assume that $\text{Area}_m(D)$ converges to an element of $\mathbb{R} \cup \{+\infty\}$ as $m \to \infty$, and our goal is to prove that this limit equals $\pi = \text{Area}(D)$.

The total mass of the measures μ_m is uniformly bounded, by (81). Lemma 7.4 implies that the support of μ_m, which is contained in $\partial K_m \times \partial K_m$, is uniformly bounded in $\mathbb{R}^2$. We may thus pass to a subsequence and assume that

$$\mu_m \xrightarrow{m\to\infty} \tilde{\mu} \tag{86}$$

weakly for some measure $\tilde{\mu}$. This means that for any continuous test function φ on $\mathbb{R}^2 \times \mathbb{R}^2$ we have $\int \varphi d\mu_m \longrightarrow \int \varphi d\tilde{\mu}$. The measure $\tilde{\mu}$ is supported on $\partial K \times \partial K$, by Lemma 7.4.

We claim that for any convex polygon $P \subseteq \mathbb{R}^2$ with $\overline{P} \subseteq D(0,2) \subseteq K$,

$$\tilde{\mu}(\mathcal{F}(P)) = \mu(\mathcal{F}(P)). \tag{87}$$

Since $\mu(\mathcal{F}(P)) = 2 \cdot \text{Perimeter}(P)$ is continuous in P and monotone in P under inclusion, and since $\mathcal{F}(P') \subseteq \mathcal{F}(P)$ when $P' \subseteq P$, in order to prove (87) it suffices to prove the following: For any two convex polygons $P_0, P_1 \subseteq \mathbb{R}^2$ with $\overline{P_0} \subseteq P_1$ and $\overline{P_1} \subseteq D(0,2) \subset K$,

$$\tilde{\mu}(\mathcal{F}(P_0)) \leq \mu(\mathcal{F}(P_1)) \qquad \text{and} \qquad \mu(\mathcal{F}(P_0)) \leq \tilde{\mu}(\overline{\mathcal{F}(P_1)}). \tag{88}$$

Now (88) follows from (86) and Lemma 7.6, and hence (87) is proven. The map T_K^{-1} is a well-defined map from $A = \{(x,y) \in \partial K \times \partial K \,;\, x \neq y\}$ to $\mathcal{G}$. By (87), the push-forward of $\tilde{\mu}|_A$ under the map T_K^{-1} is a measure $\tilde{\sigma}$ on $\mathcal{G}$ which satisfies the assumptions of Lemma 7.2. From the conclusion of Lemma 7.2,

$$\frac{1}{2\pi} \int_{\partial K \times \partial K} \text{Length}((x,y) \cap D) d\tilde{\mu}(x,y) = \frac{1}{2\pi} \int_A \text{Length}((x,y) \cap D) d\tilde{\mu}(x,y) = \pi. \tag{89}$$

By the Santaló formula,

$$\text{Area}_m(D_m^{\pm}) = \frac{1}{2\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \text{Length}_m((x,y)_m \cap D_m^{\pm}) d\mu_m(x,y).$$

We thus deduce from (80), (86) and (89) that

$$\lim_{m \to \infty} \text{Area}_m(D_m^{\pm}) = \frac{1}{2\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \text{Length}((x,y) \cap D) d\tilde{\mu}(x,y) = \pi. \tag{90}$$

However, $D_m^- \subseteq D \subseteq D_m^+$. Hence (90) implies that $\text{Area}_m(D) \longrightarrow \pi$. □

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