Question 8: Prove that any odd smooth map $f : S^n \to S^n$ has an odd degree.

Solution: By induction on the dimension n. Pick a regular value $P \in S^n$ of the function f, which exists by Sard's theorem. After a rotation we may assume that P is the north pole. The preimage $f^{-1}(P)$ is a finite set $\{P_1, \ldots, P_k\}$ in S^n, and after rotation we may assume that this finite set does not intersect the equator S^{n-1}. Since f is an odd map, also $f^{-1}(Q) \cap S^{n-1} = \emptyset$ where $Q = -P$. Pick a sufficiently small neighborhood U of the north pole $P \in S^n$, say a small cap. Then $f^{-1}(U) = U_1 \cup \ldots \cup U_k$, where the open sets $U_1, \ldots, U_k$ are disjoint and their closures do not intersect the equator. The map $f : U_i \to U$ is a diffeomorphism with $f(\partial U_i) = \partial U$ and $\deg(f|_{\partial U_i}) = \pm 1$. Thus,

$$\deg(f) = \sum_{i=1}^{k} \deg(f|_{\partial U_i}) = k \qquad \mod 2. \tag{1}$$

Set $V = -U$ and $V_i = -U_i$. A meridian in S^n is a great circle that passes through the north pole P. By moving points along the meridian towards the equator, we see that the identity map Id on S^n is smoothly homotopic to an odd map $h : S^n \to S^n$ with $h(P) = P$, $h|_{S^{n-1}} = \mathrm{Id}$ and

$$h(S^n \setminus (U \cup V)) \subseteq S^{n-1}. \tag{2}$$

The map f is therefore smoothly homotopic to the odd map $g := h \circ f$. Moreover, the degree of $f|_{\partial U_i}$ is equal to the degree of $g|_{\partial U_i}$ for all i, since the two points $f(x) \in \partial U$ and $g(x) \in S^{n-1}$ belong to the same meridian for all $x \in \partial U_i$. In fact, $g|_{\partial U_i} = R \circ f|_{\partial U_i}$, where $R : \partial U \to S^{n-1}$ is the meridian-projection to the equator, which is a diffeomorphism. Thus $\deg(g|_{\partial U_i}) = \pm 1$.

Write S^n_+ for the northern hemisphere, an open set. Denote

$$M = S^n_+ \setminus \bigcup_{i=1}^{k} [U_i \cup V_i].$$

Then $g = h \circ f : M \to S^n$ is actually a map into S^{n-1} according to (2), since $f(M)$ is contained in $S^n \setminus (U \cap V)$. As we saw in class, Stokes theorem implies that $g : \partial M \to S^{n-1}$ is a map of degree zero, since for any top form ω of non-zero integral on S^{n-1},

$$\deg(g|_{\partial M}) \cdot \int_{S^{n-1}} \omega = \int_{\partial M} g^* \omega = \int_M d(g^* \omega) = \int_M g^*(d\omega) = 0$$

as $d\omega$ vanishes on S^{n-1}, being the differential of a top form. The boundary ∂M consists of several connected components: The equator S^{n-1}, as well as all of the ∂U_i's and ∂V_i's that are contained in S^n_+. Hence,

$$0 = \deg(g|_{\partial M}) = \deg(g|_{S^{n-1}}) + \sum_{i; U_i \subseteq S^n_+} \deg(g|_{\partial U_i}) + \sum_{i; V_i \subseteq S^n_+} \deg(g|_{\partial V_i}). \tag{3}$$

Note that for any $i \in \{1, \ldots, k\}$, exactly one of the two sets U_i and V_i is contained in S^n_+. Recall that $\deg(g|_{\partial V_i}) = \pm 1$ and $\deg(g|_{\partial U_i}) = \pm 1$ for all i. Then from (1) and (3),

$$0 = \deg(g|_{S^{n-1}}) + k = \deg(g|_{S^{n-1}}) - \deg(f) \qquad \mod 2.$$

By the induction hypothesis, $\deg(g|_{S^{n-1}})$ is odd, since $g : S^{n-1} \to S^{n-1}$ is odd. Hence $\deg(f)$ is odd, as promised. The base case of the induction can either be the case $n = 1$, or (better) a trivial statement in $n = 0$, if interpreted correctly.

Question 9: Prove Wirtinger's inequality in $\mathbb{C}^2$.

This question was solved correctly by several students, but I have a small comment, which may or may not be useful. The tangent space of M at a point p is a real 2-dimensional subspace of $\mathbb{C}^2 \cong \mathbb{R}^4$, denoted by E. The real subspace E may or may not be a complex 1-dimensional subspace of $\mathbb{C}^2$. The required inequality is equivalent to the assertion that for any $\mathbb{R}$-linearly-independent vectors $u, v \in E \subseteq \mathbb{C}^2$,

$$\omega(u, v) \leq \mathrm{Area}(u, v) \tag{4}$$

where $\mathrm{Area}(u, v)$ is the area of the parallelogram spanned by $u, v \in \mathbb{C}^2$, with equality iff the real span of u and v is a 1-dimensional complex subspace. Here,

$$\omega = \frac{i(dz \wedge d\bar{z} + dw \wedge d\bar{w})}{2} = dx \wedge dy + du \wedge dv$$

where $z = x + iy$ and $w = u + iv$ and $x, y, u, v \in \mathbb{R}$ are real coordinates in $\mathbb{C}^2 \cong \mathbb{R}^4$. Verify that for any $u, v \in \mathbb{R}^4$,

$$\omega(u, -iv) = \langle u, v \rangle,$$

where $\langle u, v \rangle$ is the standard scalar product between $u, v \in \mathbb{R}^4$.

Neither the left-hand side nor the right-hand side of (4) are affected if we replace u by $u + \lambda v$ for any $\lambda \in \mathbb{R}$. We may apply such a replacement, and assume that $u \perp v$. This is the first step of the Gram-Schmidt orthogonalization process. To summarize, it suffices to show that for any orthogonal vectors $u, v \in \mathbb{C}^2 \cong \mathbb{R}^4$,

$$\langle u, iv \rangle = \omega(u, v) \leq Area(u, v) = |u| \cdot |v|.$$

This inequality follows from the Cauchy-Schwartz inequality. There is equality in the Cauchy-Schwartz inequality if and only if iv is real-proportional to u. This happens if and only if the real span of the orthogonal vectors u and v is a one-dimensional complex subspace.