Solution to Exercise 1-2b

Let $X_{u,v}$ denote the number of 2-paths u to v, and let C be the number of 4-cycles, we have:

$$4C = \sum_{\substack{v,u\in V\\ v\neq u}} \binom{X_{u,v}}{2} = \sum_{v,u\in V} \binom{X_{u,v}}{2} - \sum_{v\in V} \binom{X_{v,v}}{2} = \sum_{v,u\in V} \binom{X_{u,v}}{2} - \sum_{v\in V} \binom{deg(v)}{2} .$$

Thus

$$8C = \sum_{v,u} X_{u,v}^2 - \sum_{v,u} X_{u,v} - \sum_{w} (deg(w))^2 + \sum_{w} deg(w) . \tag{1}$$

Letting $S = \sum_{v,u} X_{u,v}$ and applying Cauchy Schwartz, we have

$$\begin{aligned} S &= \sum_{v,u,w} 1_{w\sim u \wedge w\sim v} = \sum_{w}\sum_{v,u} 1_{w\sim u}\cdot 1_{w\sim v} = \sum_{w}(\sum_{v} 1_{w\sim v})^2 = \sum_{w}(deg(w))^2 \\ &\geq \frac{(\sum_w deg(w))^2}{n} = n\cdot\Delta^2 . \end{aligned}$$

Plugging this into Eq. (1) and applying Cauchy Schwartz, we obtain

$$8C \geq \frac{S^2}{n^2} - S - S = S\left(\frac{S}{n^2} - 2\right) \geq n\cdot\Delta^2\left(\frac{\Delta^2}{n} - 2\right) = \Omega(\Delta^4) .$$