**Distributed Graph Algorithms** **Spring 2021**

# Exercise 2: May 19

*Lecturer: Merav Parter*

**Exercise 1 (Randomized Symmetry Breaking).** For an $n$-vertex graph $G = (V, E)$, and a vertex $u \in V$, let $N(u)$ be the neighbors of $u$ in $G$, and let $deg(u) = |N(u)|$ be its degree. For integers $1 \leq \alpha \leq \beta$, a subset $V' \subseteq V$ is an $(\alpha, \beta)$ ruling-set if (i) $d_G(u, v) \geq \alpha$ for every $u, v \in V'$, and (ii) for every $u \in V$, there exists $v \in V'$ such that $d_G(u, v) \leq \beta$.

Consider the following 1-round randomized LOCAL algorithm: each vertex $v$ picks a number $r_v$ u.a.r in $[0, 1]$, and joins a set $R$ if it is a local minima, i.e., if $r_v < r_u$ for any neighbor $u \in N(v)$. We next prove two properties of this algorithm.

(1a) Show that for each $v$ it holds that $R \cap N^+(v) \neq \emptyset$ with probability at least $(deg(v)+1)/(deg(v)+deg_{max})$, where $N^+(v) = N(v) \cup \{v\}$ and $deg_{max}$ is the maximum degree among nodes in $N^+(v)$.
(1b) Assume that $G$ is a nearly regular graph with all degrees in $[\Delta, 2\Delta]$. Show that w.h.p. $R$ is a $(2, O(\log n))$ ruling set. **Bonus**: prove the above claim for any $n$-vertex graph $G$, i.e., without assuming that $G$ is nearly regular.

**Exercise 2 (Strong Network Decomposition).** Let $\mathcal{A}$ be the deterministic LOCAL algorithm for $(c, d)$ *weak* network decomposition of [RG20], and let $\mathcal{B}$ be the deterministic sequential algorithm for $(c, d)$ *strong* network decomposition where $c, d = O(\log n)$. Both of these algorithms were presented in class. In this exercise, we will use algorithms $\mathcal{A}$ and $\mathcal{B}$ (in a black box manner) in order to compute strong network decomposition in the LOCAL model.

**Show**: There is a deterministic LOCAL algorithm for computing for computing $(O(\log n), O(\log n))$ strong network decomposition using $poly(\log n)$ rounds. Hint: focus on the construction of the first color class in the desired strong network decomposition. Apply algorithm $\mathcal{A}$ on the power graph[1] $G^q$ for some parameter $q$ to be set carefully, and let the nodes locally apply the sequential algorithm $\mathcal{B}$ on the output clusters of $\mathcal{A}$.

**Exercise 3 (Low Diameter Ordering).** The last exercise illustrates another combinatorial application of network decomposition which also sets a convenient platform for simulating SLOCAL (i.e., sequential local) algorithms in the LOCAL model.

**Definition 2.1** *Given a $n$-node graph $G = (V, E)$, a $d(n)$-diameter ordering of $G$ is an assignment of unique labels[2] to all nodes $V$ such that any path $P$ on which the labels are increasing along $P$, any two nodes of $P$ are within a distance $d(n)$ in the graph $G$.*

(2a) Show that any $n$-vertex graph has a $d(n)$-diameter ordering with $d(n) = O(\log^2 n)$.
(2b) Show that given that all nodes are provided with $d(n)$-ordering w.r.t. the graph $G^r$, then any SLOCAL algorithm with locality $r$ can be simulated in $O(r \cdot d(n))$ LOCAL rounds.

[1] Recall that in the power graph $G^q$, $(u, v) \in E(G^q)$ iff $d_G(u, v) \leq q$.
[2] The label is simply a unique identifier to the vertices of $O(\log n)$ bits, i.e., a number in $[1, poly(n)]$.