Network Decomposition Alg'

Scribe: Avi Cohen

Lecture 5
5/5/2021

Strong (c,d) ND: Given graph $G=(V,E)$, a (c,d) ND is a partitioning into vertex disjoint subgraphs $G_1,\dots,G_c$ s.t. diam of every conn. comp. of G_i is at most d.

Path may go outside of the vertices of X_{ij}

Weak (c,d) ND: $\text{w-diam}(X_{ij}) = \max_{u,v \in V(X_{ij})} d_{G}(u,v)$

cluster of G_i

They do no have to induce connected comp.

Each G_i is a union of clusters $\underbrace{X_{i1},\dots,X_{i\ell}}_{\text{vertex disjoint non-neighboring}}$ s.t. $\text{w-diam}(X_{ij}) \le d$.

Note: In LOCAL model weak ND is good enough. In CONGEST we need strong ND.

Thm [AGL'89]: Every n-vertex graph $G=(V,E)$ has (c,d)-ND with $c,d=O(\log n)$

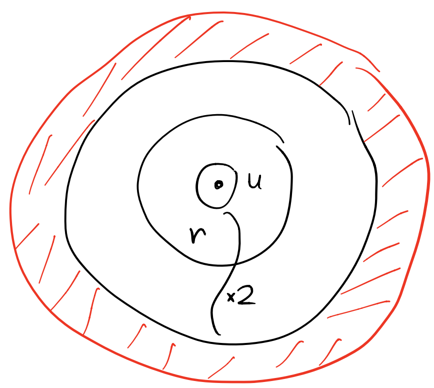
[Tight!]

Goal: $V' \subseteq V$ s.t.

① $|V'| > \frac{|V|}{2}$

② V' is made of clusters of weak/strong diam $O(\log n)$.

Algorithm for computing one color class



$G' \leftarrow G$, $V' \leftarrow \emptyset$

while G' is non-empty:

* Pick $u \in G'$ and grow a ball around u in G' upto the min. r s.t.:

$$|B_G(u,r+1)| < 2\,|B_{G'}(u,r)|$$

* Add $B_{G'}(u,r)$ to V'
* $G' \leftarrow G' \setminus B_{G'}(u,r+1)$

Note that we add to V' more vertices than we throw away.

Lemma 1: $|V'| \geq \frac{|V|}{2}$ ✓

Lemma 2: diam of each conn. comp. in $G[V']$ is at most $O(\log n)$.

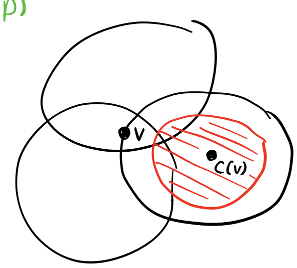
How can we do this faster?

Thm [LS'93]: For every n-vertex graph $G=(V,E)$, there is a randomized LOCAL alg. for computing weak (c,d) ND where $c,d = O(\log n)$ with $O(\log^2 n)$ rounds w.h.p.

Algorithm for computing one color class

* Every vertex u picks $r_u \sim Geo(p = 1/2)$ $\quad Pr[r_u = y] = p \cdot (1-p)^{y-1}$
* Send (u, r_u) to all nodes in $B_G(u, r_u)$
* Every v defines $c(v)$ to be: $\quad$ cluster center.

$$c(v) = \operatorname{argmin}\{ ID(u) \mid r_u \geq d_G(u,v) \}$$



* v becomes unclustered if $d_G(v, c(v)) = r_{c(v)}$. $\quad$ makes the balls disjoint.

V' = collection of all clustered vertices.

Note that it is possible for v not to be in its own cluster even if the cluster is not empty. $\Rightarrow$ weak ND.

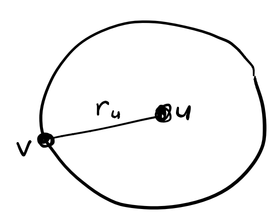
Lemma 1: Diam of each cluster is $O(\log n)$.

↳ follows from properties of Geo distribution + Union Bound.

Lemma 2: $|V'| \geq \frac{|V|}{2}$

Pf: We will show: $Pr[v \text{ is unclustered}] = p \ (= 1/2)$.

$$Pr[v \text{ is unclustered}] = \sum_{u \in V} \underbrace{Pr[v \text{ unclustered} \mid c(v) = u]}_{=p} \cdot Pr[c(v) = u]$$



Define the following events.

D_u: $r_u = d_G(u,v)$

E_u: $r_u \geq d_G(u,v)$

F_u: $\forall u' < u,\ r_{u'} < d_G(u', v)$

$\Pr[v \text{ unclustered} \mid c(v)=u] =$

D_u / E_u and F_u are independent.

$$\frac{\Pr[v \text{ unclustered and } c(v)=u]}{\Pr[c(v)=u]} = \frac{\Pr[D_u \text{ and } F_u]}{\Pr[E_u \text{ and } F_u]} = \frac{\Pr[D_u]}{\Pr[E_u]} = p$$

$\square$

Deterministic LOCAL ND

Thm [RG '20]: For every $G=(V,E)$, there is a deterministic LOCAL algorithm for computing weak (c,d) ND with $c=O(\log n)$, $d=O(\log^3 n)$ within $O(\log^7 n)$ rounds.

This result was a breakthrough! Previous state of the art was $2^{\sqrt{\log n}}$ rounds.

Algorithm for computing one color class

* Assume every node has $b=O(\log n)$-bit ID.
* Node u has $\ell(u)$. label of u.
* cluster: nodes with the same label.

set of "live" nodes after phase i

$$V' = V_b \subseteq \ldots\ldots \subseteq V_2 \subseteq V_1 \subseteq V_0 = V$$

Invariant for the beginning of phase $i \in \{0,\ldots,b-1\}$

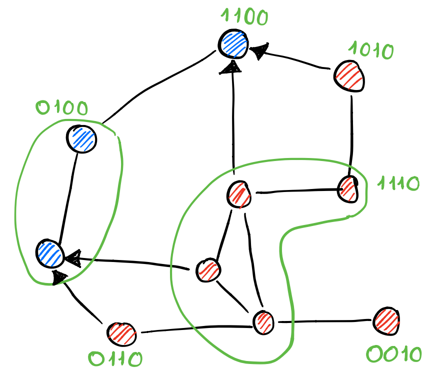
1) Conn. comp. of each $G[V_i]$ agree on i-length suffix of their label.
2) Diam in G of cluster is at most $i \cdot R$. $O(b\cdot\log n)$
3) $|V_{i+1}| \geq (1-\frac{1}{2b})|V_i|$

$\Rightarrow$ After b phases, every conn. comp. is a cluster of diam $O(\log^3 n)$.

Additionally, $|V'| \geq (1-\frac{1}{2b})^b |V|$.

Phase i

Conn. Comp of $G[V_i]$, $\ell(u) = [****\underbrace{Y}_{i \text{ bits}}]$



Red vertices $[***1Y]$

Blue vertices $[***0Y]$

To get to the next phase suffices to separate red and blue verts. Blue verts will remain clustered, Red will convert to blue or die.

Step j of Phase i:

Repeat $R = 4b \cdot \log n$ Steps

* Every red node sends req. to join one arbitrary neighboring blue cluster.
* Every blue cluster A has two options:
  1) $\# \text{req} \geq \frac{|A|}{2b}$: accept all req. and the red req. nodes become blue and join cluster.
  2) $\# \text{req.} < \frac{|A|}{2b}$: omit all requests, red req. nodes die and stop grow.

 # Rounds: $\underbrace{\log n}_{\#\text{color classes}} \times \underbrace{\log n}_{\#\text{phases}} \times \underbrace{\log^2 n}_{\#\text{steps in phase.}} \times \underbrace{\log^3 n}_{\text{one step}}$

Lemma 1: After $4b \log n$ steps, all blue clusters stop growing.

Pf: $\left(1 + \frac{1}{2b}\right)^{4b \log n} > n$

Lemma 2: Once a blue cluster stops, it has no red neighbors.

Lemma 3: Diam cluster $\leq (i+1) \cdot R$

Lemma 4: $|V_{i+1}| \geq \left(1 - \frac{1}{2b}\right) |V_i|$