Succinct Graph Structures and Their Applications **Spring 2020**

Home Exam : Aug 4

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Instructions: The exam consists of 4 problem sets, each worth 25 points. In case needed (and unless stated otherwise), you are allowed to use constructions that we showed in class as a black-box, but still required to add full proofs. Typing in LaTeX (and such) is highly recommended, but not a must (in case you do not have time). Please submit your solutions through email to merav.parter@weizmann.ac.il by midnight of Aug 5, 12am. Good luck!

Labeling Schemes

Exercise 1. A separator of a graph G is a subset of vertices S whose removal breaks $G \setminus S$ into components of size at most $2/3 \cdot n$. In this exercise, we design short labels for trees based on small separators.
(a) Show that every n-vertex tree T has a separator of size 1, i.e., there exists a vertex v whose removal breaks T into forests, each of size at most $2/3 \cdot n$.
(b) show an $O(\log^2 n)$-size labeling scheme for computing the LCA (lowest common ancestor) between two vertices u and v. I.e., design labels $L(u)$ of $O(\log^2 n)$ bits such that given $L(u)$ and $L(v)$, one can compute the LCA of u and v in the tree T.
(c) Let T be a weighted tree with edge weights in $[1, W]$. Show an $O(\log^2 n + \log n \cdot \log W)$-size labeling scheme for computing the largest edge weight on the tree path between u and v. I.e., such that given $L(u)$ and $L(v)$, one can compute the weight of the heaviest edge on the path between u and v in the tree. Remark: You are allowed to use any construction that you want, even if it does not use the notion of separators.

Spanners

Exercise 2. We showed in class the construction of $(2k-1)$ spanners with $O(n^{1+1/k})$ edges. This size-stretch tradeoff is believed to be tight based on the Erdős girth conjecture. The given exercise demonstrates that graphs with large minimum degree admit considerably sparser spanners. (a) Show that every n-vertex graph G with minimum degree Δ has a 5-spanner with $\widetilde{O}((n/\Delta)^2)$ edges. (b) Generalize this result to show that it has an $(6k-1)$-spanner with $\widetilde{O}((n/\Delta)^{1+1/k})$ edges. Hint: consider the cluster-graph induced on n/Δ cluster centers.

Tree Embedding

Exercise 3. In this exercise we will consider a deterministic procedure for computing a low-diameter decomposition. The benefit of this procedure is that it also handles multi-graphs (where a given edge might have several copies in the graph). In addition, the clusters computed by this procedure will have small *strong-diameter*[1]. Let $c(e)$ be the number of copies of an edge $e \in G$ and for a subset of edges $F \subseteq E(G)$, let $C(F) = \sum_{e \in F} c(F)$. Let $E(u,r) = \{(x,y) \in E(G) \mid x, y \in B(u,r)\}$ be the G-edges connecting vertices in $B(u,r)$. The input to the decomposition algorithm $\mathsf{DetDecomp}$ (see Fig. 0.1) consists of (1) a multi-graph $G = (V, E, c)$ (where each edge $e \in E$ has $c(e)$ copies in G), and (2) a desired diameter parameter D. Consider an unweighted undirected graph $G = (V, E)$ with $C = C(E)$ and let $\alpha = 4\ln(C)/D$. Show that Alg. $\mathsf{DetDecomp}(G, D, \alpha)$ returns subsets $V_1, \ldots, V_k$ such that:

(Q1) The strong diameter of each subgraph $G[V_i]$ is at most D.

(Q2) There are at most $\alpha \cdot C(E)$ inter-cluster edges (i.e., edges connecting $u \in V_i$ and $v \in V_{j \neq i}$).

[1]The strong diameter of a subgraph $G' \subseteq G$ is $\max_{u,v \in G'} \mathtt{dist}_{G'}(u,v)$.

Algorithm DetDecomp($G = (V, E, c), D, \alpha$)

1. Set $\ell \leftarrow 1$.
2. While G is nonempty do:
   (a) Pick a vertex v in G.
   (b) Let r_v be the smallest r satisfying that $C(E(v, r+1)) \leq (1+\alpha)C(E(v, r))$.
   (c) $V_\ell \leftarrow B(v, r_v)$.
   (d) $\ell \leftarrow \ell + 1$.
   (e) Remove all vertices of V_ℓ from G (along with their edges).
3. Return $V_1, \ldots, V_k$.

Figure 0.1: Deterministic low-diameter decomposition algorithm

Fault Tolerant Graph Structures

Exercise 4. For an undirected graph G, a subgraph $G' \subseteq G$ and vertex pair $u, v \in V$, define $\texttt{conn}(u, v, G') = 1$ iff u and v are connected in G'. A subgraph $H \subseteq G$ is an *f-edge FT-connected* subgraph if for every $u, v \in V$ and $F \subseteq E$, $|F| \leq f$, it holds that:

$$\texttt{conn}(u, v, H \setminus F) = \texttt{conn}(u, v, G \setminus F) \ .$$

f-vertex FT-connected subgraphs are defined analogously only with $F \subseteq V$. (a) Show an $O(f \cdot m)$-time algorithm that given any n-vertex graph G and parameter f, constructs an *f-edge FT-connected* subgraph $H \subseteq G$ with $O(f \cdot n)$ edges. (b) Show that every n-vertex graph has an *f-vertex FT-connected* subgraph H with $O(f^2 \cdot n \cdot poly(\log n))$ edges.