Distance Oracles

We presented in class a \((2k - 1)\) approximate distance oracle scheme by Thorup and Zwick [TZ05]. In this exercise, we will provide a tighter analysis for this scheme in certain settings.

**Exercise 1.** We analyzed the query algorithm and showed that given an arbitrary query pair \(u, v\), the output distance estimate \(\hat{d}(u, v)\) can be bounded by 
\[\hat{d}(u, v) \leq (2k - 1) \cdot \text{dist}(u, v, G).\]
We are now interested in whether a tighter analysis of the stretch can be obtained for certain query pairs. Suppose that the query pair \(u, v\) is such that \(u \in A_i\) for \(i \geq 1\). Prove or Refute: for such a query, the query algorithm returns a distance approximation which is strictly better than \(2k - 1\). In particular, bound the stretch factor \(f(k, i)\) such that 
\[\hat{d}(u, v) \leq f(k, i) \cdot \text{dist}(u, v, G),\]
for \(u \in A_i\).

**Exercise 2.** The distance oracle presented in class supported distance queries \(u, v\) for any \(u, v \in V \times V\). Our goal is to construct a smaller oracle of size \(o(n^{1+1/k})\) that would handle only a subset of query pairs. In particular, you are now given a subset of vertices, called hereafter, sources, \(S \subseteq V\) and it is required to design a source-wise approximate distance oracle scheme. In such a scheme, the query algorithm receives only queries of the form \(u, v \in S \times V\) (i.e., your oracle should only answer distance queries between one of the sources and some other vertex in the graph). Show that in such a case, the preprocessing algorithm can be adapted to yield a data structure of size \(\tilde{O}(k|S|^{1/k} \cdot n)\). Your answer should include a modified preprocessing algorithm, query algorithm and a correctness analysis. Hint: Modify the definition of \(A_0\) and the sampling probabilities.

**References**