Distributed Graph Algorithms	**Spring 2021**

Exercise 4 (Due July 12)

Lecturer: Merav Parter

Exercise 1: Multiplicative Spanners. Show that every n-vertex graph G with minimum degree $\sqrt{n}$ admits a 5-spanner $H \subseteq G$ with $O(n \log^2 n)$ edges. In addition, show that such a spanner can be computed in $O(1)$ rounds in the LOCAL model. The algorithm can be *randomized*, and it is required to output a 5-spanner with the desired number of edges, with high probability.

Exercise 2: Additive Spanners. Consider a D-diameter n-vertex graph $G = (V, E)$, and let $S \subset V$ be a random sample of $O(\sqrt{n} \log n)$ vertices. In this exercise, we compute a 2-additive spanner H for G obtained by taking the union of two subsets of edges: T_S and H_{low}. The set T_S is a union of $|S|$ BFS trees rooted at each source $s \in S$. The set H_{low} consists of all edges incident to low-degree vertices, namely, vertices with degree at most $\sqrt{n}$. Formally,

$$T_S = \bigcup_{s \in S} BFS(s) \text{ and } H_{low} = \{(u, v) \in E(G) \mid deg(u) \leq \sqrt{n}\} .$$

(i) Show that $H = T_S \cup H_{low}$ is a 2-additive spanner. I.e., H should satisfy that

$$\texttt{dist}_H(u, v) \leq \texttt{dist}_G(u, v) + 2, \forall u, v \in V .$$

Hint: fix a pair u, v and consider its shortest path P in G. Zoom into one specific edge on $P \setminus H_{low}$.
(ii) Provide a randomized CONGEST algorithm for computing H in $\widetilde{O}(D + \sqrt{n})$ rounds (w.h.p.). You may use theorems shown in class in a black-box manner, but still required to formally state them.

Exercise 3: Low Congestion Shortcuts. Given is an n-vertex graph $G = (V, E)$ and a collection of vertex-disjoint subsets $S_1, \ldots, S_N$ where $G[S_i]$ (i.e., the induced subgraph) is connected for every S_i. Recall that (α, β) shortcuts for these sets is collection of subgraphs $H_1, \ldots, H_N$ that satisfies the following: (1) the diameter of each subgraph $G[S_i] \cup H_i$ is at most α and (2) each edge $e \in G$ appears on at most β subgraphs. Show that every graph with minimum degree k has (α, β) shortcuts with $\alpha = O(n/k)$ and $\beta = 2$.