Spanners

Graph spanners introduced by Peleg and Schäffer [PS89] are sparse subgraphs that faithfully preserve the distances in the original graph up to some bounded stretch. Formally, for a given undirected $n$-vertex graph $G = (V, E)$, a subgraph $H \subseteq G$ is an $(\alpha, \beta)$-spanner for $G$ if for every $u, v \in V$, it holds:

$$\text{dist}(u, v, H) \leq \alpha \cdot \text{dist}(u, v, G) + \beta.$$ 

When $\beta = 0$, the spanner is called multiplicative and when $\alpha = 1$, the spanner is additive. In this class, we mainly consider multiplicative spanners, which we usually denote by $(2^k - 1)$-spanners, i.e., spanners with multiplicative stretch $2^k - 1$. As a warm-up example, note that every $n$-clique has a 2-spanner with $n - 1$ edges, simply by taking all edges incident to one vertex. There are, however, graphs for which any 2-spanner has $\Omega(n^2)$ edges (e.g., the complete bipartite graph).

**Greedy Construction.** We will now see a very simple greedy algorithm due to [ADD+93] that constructs $(2^k - 1)$ spanners for (possibly) weighted graphs with $O(n^{1+1/k})$ edges. For simplicity assume that the edge weights in the graph are unique, $w(e_i) \neq w(e_j)$. For a weighted graph $G$ and $u, v \in V(G)$, $\text{dist}(u, v, G)$ is the weight of the shortest-path between $u$ and $v$ in $G$. See Fig. 1.2 for the description of the algorithm.

Next, we analyze this construction and begin by showing that $H$ is a $(2^k - 1)$-spanner for $G$.

**Algorithm GreedySpanner($G$)**

1. Consider $G$ edges in increasing weight ordering $w(e_1) < \ldots < w(e_m)$.
2. $H \leftarrow \emptyset$.
3. For $i = 1, \ldots, m$:
   (a) Add the edge $e_i = (u_i, v_i)$ to $H$ only if: $\text{dist}(u_i, v_i, H) > (2k - 1)\text{dist}(u_i, v_i, G)$.
4. Output $H$.

![Figure 1.1: The Greedy Algorithm for Constructing $(2k - 1)$ Spanners](image)

**Lemma 1.1 (Stretch)** For every $u, v \in V$, $\text{dist}(u, v, H) \leq (2k - 1) \cdot \text{dist}(u, v, G)$.

**Proof:** By construction, the lemma holds for every $(u, v) \in E$. Now, consider an arbitrary pair $x, y \in V$ and let $P$ the $x$-$y$ shortest path in $G$. Since for each edge $e = (u, v) \in P$, there is a $u$-$v$ path in $H$ of length at most $(2k - 1)w(e)$, it holds that $\text{dist}(u, v, H) \leq (2k - 1) \sum_{e \in P} w(e) = (2k - 1)\text{dist}(u, v, G)$. The fact that it is sufficient to bound the stretch between neighboring vertex pairs is a very convenient property of multiplicative spanners. It does not hold for additive spanners, which are indeed much more intriguing structures as we will see in later sessions.

We will bound the size of the spanner in two steps. An important definition in this context is the girth of the graph, namely, the length of the shortest cycle in $G$. We first show that the output spanner of Alg. GreedySpanner($G$) has a large girth and then show that every graph of large girth must be sparse.

**Claim 1.2** The girth of $H$ is at least $2k + 1$. 

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1-1
Proof: Assume towards contradiction that $H$ has a cycle $C$ of length at most $2k$. Let $e = (u,v) \in C$ be the cycle edge that was added last to $H$ (among all other $C$ edges). By the addition of $e$, we have that $\sum_{e' \in C \setminus \{e\}} w(e') > (2k - 1)w(e)$. On the other hand, by the fact that $C \setminus \{e\}$ contains at most $2k - 1$ edges that are lighter than $e$, it holds that $\sum_{e' \in C \setminus \{e\}} w(e') < (2k - 1)w(e)$, leading to contradiction. ■

Claim 1.3 Every $n$-vertex graph $G$ with girth $g(G) \geq 2k + 1$ has $O(n^{1+1/k})$ edges.

Proof Sketch: Assume towards contradiction that there exists an $n$-vertex graph $G$ with at least $4n^{1+1/k}$ edges, and girth at least $2k + 2$. We first compute a subgraph $G' \subseteq G$ with at least $2n^{1+1/k}$ edges and minimum degree $2n^{1/k}$. To compute $G'$, we repeatedly omit from $G$ vertices with degrees less than $2n^{1/k}$, one by one. Clearly, the girth of $G'$ is also at least $2k + 2$ (by the contradictory assumption and as $G' \subseteq G$).

We now consider some vertex $u \in G'$ and bound the number of vertices at distance at most $k$ from it in $G'$. The key observation is that since the girth of $G'$ is at least $2k + 1$, two vertices $x,y$ at distance $i \leq k - 1$ from $u$ have no common neighbor. Hence, the number of vertices at distance $i$ from $u$ is at least $(2n^{1/k})^i$ for every $i \leq k - 1$. Since pair of vertices at distance $k - 1$ from $u$ have no common neighbor, the number of vertices at distance $k$ from $u$ is at least $(2n^{1/k})^k > n$, leading to contradiction.

Erdős’ Girth Conjecture. A well-known conjecture by Erdős states that for every $k \geq 1$ and for sufficiently large $n$, there are $n$-vertex graphs with girth at least $2k + 2$ and $\Omega(n^{1+1/k})$ edges. The conjecture was verified for $k = 1,2,3,5$. Consider the $n$-vertex graph $G$ with girth at least $2k + 2$ and $\Omega(n^{1+1/k})$ edges which is promised to exist by the conjecture. A removal of any edge $(u,v)$ from $G$ increases the distance between $u,v$ from 1 to $2k + 1$ and hence any $(2k - 1)$-spanner for $G$ must include all edges of $G$. This in turn implies that under the girth conjecture, the greedy construction gives a spanner of optimal size (up to constant factors).

3-Spanners via a Clustering Method

We will now see a different approach for constructing 3-spanners by Baswana and Sen [BS07]. The algorithm works also for weighted graphs (up to minor modifications). For demonstrating the clustering ideas in a clean manner, we will focus on unweighted graphs. Unlike the greedy construction, this algorithm is randomized and it guarantees to construct a 3-spanner with probability $\geq 1 - 1/n^c$ for some constant $c$. For a vertex $u \in V$, let $\Gamma(u)$ be the neighbors of $u$ in $G$ and let $\deg(u)$ denotes its degree in $G$. Call a vertex $v$ high-degree if $\deg(v,G) \geq 5\sqrt{n}\log n$, otherwise, the vertex is low-degree. Let $V_h$ be the collection of high-degree vertices and let $V_v$ be the set of all low-degree vertices. First, we add to the spanner $H$ all edges incident to low-degree vertices. To handle the high-degree vertices, we sample a set of $O(\sqrt{n})$ centers $S \subseteq V$ by sampling each vertex $v \in V$ independently with probability $1/\sqrt{n}$ and adding it to $S$. Each center $s \in S$ would correspond to a cluster that consists of a subset of its high-degree neighbors. These clusters are defined by letting each high-degree vertex joins the cluster of one of its neighboring centers, as a result, we have $|S| = O(\sqrt{n})$ star-clusters in $H$. Finally, for each $v \in V_h$ and for each cluster $C(s)$ centered at $s$, we add one edge between $v$ and $C(s)$ (if exists) to $H$. We next analyze the construction.

Observation 1.4 Each high-degree vertex $v$ has a neighbor in $S$, i.e., $\Gamma(v) \cap S \neq \emptyset$, with probability $1 - 1/n^4$.

Proof: Fix a high-degree vertex $v \in V$. The probability that none of $v$'s neighbors is sampled into $S$ is $(1 - 1/\sqrt{n})^{\deg(v)} \leq (1 - 1/\sqrt{n})^{5\log n/\sqrt{n}} \leq 1/n^5$. By doing a union bound over all high-degree vertices, the observation follows. ■

Lemma 1.5 $H \subseteq G$ is 3-spanner with $O(n^{3/2} \log n)$ edges, with high probability.

Proof: Recall that for multiplicative spanners, it is sufficient to bound the stretch for neighboring pairs. Since all edges incident to low-degree vertices are in $H$, it remains to consider an edge $(u,v)$ where $u,v \in V_h$. If $c(u) = c(v)$, both neighbors belong to the same star-cluster and hence $\text{dist}(u,v,H) \leq 2$. Otherwise, if $(u,v) \notin H$, $u$ must have another neighbor $w \in V_h$ that belongs to the same cluster of $v$ and such that $(u,w)$ was added to the spanner. Letting $s$ be the cluster center of $w$ and $v$, there is a $u$-$v$ path $u \rightarrow w \rightarrow s \rightarrow v$ in

\[^1\text{We call such a success probability, high probability.}\]
Algorithm 3Spanner($G$)

1. $E_1 = (V_t \times V) \cap E(G)$.
2. $S \leftarrow \text{Sample}(V, 1/\sqrt{n})$.
3. For each $v \in V_h$, let $c(v)$ be an arbitrary center vertex in $S \cap \Gamma(v)$.
4. $E_2 = \{(v, c(v)) \mid v \in V_h\}$.
5. For each $s \in S$, let $C(s) = \{v \in V_h \mid c(v) = s\}$.
6. For each $v \in V_h$, and $s \in S$, let $e(v, s)$ be one edge connecting $v$ and $C(s)$ if exists.
7. $E_3 = \bigcup_{v \in V_h} \bigcup_{s \in S} \{e(v, s)\}$.
8. Output $H \leftarrow E_1 \cup E_2 \cup E_3$.

Figure 1.2: The Clustering-Based Algorithm for Constructing 3-Spanners

$H$ of length 3. We next bound the size of $H$. The set $E_1$ has $O(n^{3/2} \log n)$ edges. The set $E_2$ contains $O(n)$ edges and since each high-degree vertex adds at most one each for each of the $O(\sqrt{n})$ clusters, $|E_3| = O(n^{3/2})$.

References

