Additive Spanners

A β-additive spanner $H$ of an unweighted\(^1\) graph $G = (V, E)$ is a subgraph of $G$ satisfying $\text{dist}(u, v, H) \leq \text{dist}(u, v, G) + \beta$ for every vertex pair $u, v \in V$. In contrast to multiplicative spanners where for every integer $k \geq 1$, there is a $(2k-1)$-spanner with $O(n^{1+1/k})$ edges, there are only 3 additive spanners for stretch values of $2, 4$ and $6$ of size $O(n^{3/2}), \tilde{O}(n^{7/5})$ and $O(n^{4/3})$ respectively [ACIM99, Che13, BKMP10]. Understanding the general size-stretch tradeoff for additive spanners was one of the biggest open problems in the area for the last twenty years. Recently, Abboud and Bodwin have made a quite shocking breakthrough which essentially implies that there are no new additive spanners to be revealed, unless settling for a polynomially large stretch.

**Theorem 9.1** [AB17] There is no $+n^{o(1)}$ additive spanners with $O(n^{4/3-\epsilon})$ edges, for any fixed $\epsilon$.

2-Additive Spanners. We start by showing a construction of 2-additive spanners with $\tilde{O}(n^{3/2})$ edges due to Aingworth et al. [ACIM99]. For simplicity, we present a randomized construction, however, it is easy to derandomize it using standard hitting-set tools. In this context, we say that a vertex $v$ to Aingworth et al. [ACIM99]. For simplicity, we present a randomized construction, however, it is easy to show that $H$ is a 2-additive spanner.

**Lemma 9.2** W.h.p., for every $u, v \in V$, $\text{dist}(u, v, H) \leq \text{dist}(u, v, G) + 2$.

**Proof:** Unlike multiplicative spanners, here it is not sufficient to make the stretch argument only for neighboring pairs $(u, v) \in E$, but rather we need to show it for every pair. Fix $u, v \in V$ and let $P$ be some $u-v$ shortest path in $G$. If $P \subseteq H$, we are done as $\text{dist}(u, v, H) = |P|$. Otherwise, $P$ must include at least one high-degree vertex $w$ (actually, it must have at least two!). By Chernoff bound (a hitting-set argument), w.h.p., $w$ has a neighbor $s \in N(w) \cap S$. Let $T$ be the BFS tree $T$ rooted at $s$ that was added into $H$ and let $P_1 = \pi(u, s, T)$, $P_2 = \pi(s, v, T)$ be the two shortest paths in this tree $T$. Consider the $u-v$ path $P_3 = P_1 \cup P_2$ obtained by concatenating $P_1$ and $P_2$. Since $T \subseteq H$, we have that $P_3 \subseteq H$ and thus by the triangle inequality:

$$\text{dist}(u, v, H) \leq |P_3| = |P_1| + |P_2| \leq \text{dist}(u, v, G) + 1 + \text{dist}(w, v, G) + 1 \leq \text{dist}(u, v, G) + 2.$$

The lemma follows. \(\blacksquare\)

4-Additive Spanners. We next show a construction of 4-additive spanners with $\tilde{O}(n^{7/5})$ due to Chechik [Che13]. We now call vertex $v$ high-degree if $\deg(v, G) \geq n^{2/5}$ and low-degree otherwise. Let $V_h$ be the subset of high-degree vertices. For every $u, v$, let $\pi(u, v)$ be the unique shortest-path between $u$ and $v$ in $G$ (we break shortest-path ties in a consistent manner).

1. Add to $H$ all edges incident to low-degree vertices.
2. Sample a subset $S$ of $\Theta(n^{2/5} \log n)$ vertices, by adding each $v$ into $S$ independently with probability $p_S = \Theta(\log n/n^{3/5})$. Add to $H$, a BFS tree of each $s \in S$.

\(^1\)Unlike multiplicative spanners, additive spanners are interesting only for the unweighted case.
(3) Sample a subset $T$ of $\Theta(n^{3/5} \log n)$ vertices, by adding each $v$ into $T$ independently with probability $p_r = \Theta(\log n/n^{2/3})$. For each $v \in V_h$, let $s(v) \in N(v) \cap T$ and for every $t \in T$, let $B(t) = \{v \in V_h \mid s(v) = t\}$. Add the edges $\{(v, s(v)) \mid v \in V_h\}$ to $H$.

(4) For each $t, t' \in T$, define $\Phi_{t, t'}$ by:

$$\Phi_{t, t'} = \{\pi(u, v) \mid u \in B(t), v \in B(t'), \pi(u, v) \text{ has at most } n^{1/5} \text{ high-deg vertices}\}.$$ 

(5) In each set $\Phi_{t, t'}$ for $t, t' \in T$, pick the shortest path and add it to $H$.

We next analyze the construction and start with size analysis.

**Observation 9.3 (Size)** $H$ has $\tilde{O}(n^{7/5})$ edges.

**Proof:** Steps (1,2,3) clearly add $\tilde{O}(n^{7/5})$ edges. In Step (5), for each pair $t, t'$, we add $O(n^{1/5})$ edges to $H$. Since $|T| = O(\log n \cdot n^{3/5})$, overall this adds $O(|T|^2 \cdot n^{1/5}) = \tilde{O}(n^{7/5})$ edges.

**Claim 9.4 (Stretch)** $\text{dist}(u, v, H) \leq \text{dist}(u, v, G) + 4$.

**Case 1:** $\pi(u, v)$ has more than $n^{1/5}$ high-deg vertices. We claim that the number of vertices that have a neighbor on the path $\pi(u, v)$ is $\Omega(n^{3/5})$. Formally, letting $N(\pi(u, v)) = \bigcup_{x \in \pi(u, v)} N(x)$, we will show that $|N(\pi(u, v))| = \Omega(n^{3/5})$. The claim follows by noting that each vertex can have at most 3 neighbors on a given shortest-path, and thus the neighborhood sets of $N(x), N(x')$ for $x, x' \in \pi(u, v)$ are effectively almost vertex-disjoint (i.e., each vertex is counted at most 3 times in these $N(x)$ subsets). By the hitting set argument, we get that there exists $s \in N(\pi(u, v)) \cap S$. Since we added a BFS tree rooted at $s$ to $H$, we get a stretch of $+2$ (same argument as in the 2-additive case).

**Case 2:** $\pi(u, v)$ has at most $n^{1/5}$ high-deg vertices. Let $x$ the closest high-degree vertex to $u$ on $\pi(u, v)$ and let $y$ be the closest high-degree vertex to $v$ on this path. Since $\pi(u, v)$ has at least one edge not in $H$ (otherwise, we are done), we get that $x \neq y$. Let $t = s(x)$ and $t' = s(y)$. By definition, $\pi(x, y) \in \Phi_{t, t'}$. If $\pi(x, y)$ is added in step (5), then $\pi(u, v) \subset H$, and we are done. Otherwise, in Step (5), the algorithm picks the shortest path $\pi(x', y') \in \Phi_{t, t'}$ and adds it to $H$. We now consider the following $u-v$ path $P$ in $H$ where

$$P = \pi(u, x) \circ (x, t) \circ (t, x') \circ \pi(x', y') \circ (y', t') \circ (t', y) \circ \pi(y, t).$$

Let $\ell = |\pi(x', y')|$, we have:

$$\text{dist}(u, v, H) \leq |P| = \text{dist}(u, x, G) + 2 + \ell + 2 + \text{dist}(y, v, G) \leq \text{dist}(u, v, G) + 4,$$

where the last inequality follows by the fact that $|\pi(x, y)| \geq |\pi(x', y')| = \ell$, by the selection of $\pi(x', y')$ in Step (5). The stretch argument follows.

**References**


