All pairs shortest paths in $O(nm)$ time

Abstract:

We present an $O(nm)$ time algorithm for all-pairs shortest paths computations in a directed graph with $n$ nodes, $m$ arcs, and nonnegative integer arc costs. This matches the complexity bound attained by Thorup for the all-pairs problems in undirected graphs. Our main insight is that shortest paths problems with approximately balanced directed cost functions can be solved similarly to the undirected case. Our algorithm starts with a preprocessing step that transforms the cost vector into a reduced cost vector that is approximately balanced. Using these reduced costs, every shortest path query can be solved in $O(m)$ time using an adaptation of Thorup’s component hierarchy method. The balancing result is of independent interest, and gives the best currently known approximate balancing algorithm for the problem. This research is joint with Laszlo Vegh.