Almost Optimal eps bias

The question of finding an epsilon-biased set with close to optimal support size, or, equivalently, finding an explicit binary code with distance 1/2-epsilon and rate close to the Gilbert-Varshamov bound, attracted a lot of attention in recent decades. In this paper we solve the problem almost optimally and show an explicit epsilon-biased set over k bits with support size $O(k/\epsilon^{2+o(1)})$. This improves upon all previous explicit constructions which were in the order of $k^{2+o(1)}$, $k/\epsilon^3$ or $(k/\epsilon^2)^{5/4}$. The result is close to the Gilbert-Varshamov bound which is $O(k/\epsilon^2)$ and the lower bound which is $\Omega(k/\epsilon^2 \log(1/\epsilon))$. The main technical tool we use is bias amplification with the s-wide replacement product. The sum of two independent samples from a biased set is $\epsilon^2$ biased. Rozenman and Wigderson showed how to amplify the bias more economically by choosing two samples with an expander. Based on that they suggested a recursive construction that achieves sample size $O(k/\epsilon^4)$. We show that amplification with a long random walk over the s-wide replacement product reduces the bias almost optimally.