The Gaussian Double-Bubble and Multi-Bubble Conjectures

Abstract:

The classical Gaussian isoperimetric inequality, established in the 70's independently by Sudakov-Tsirelson and Borell, states that the optimal way to decompose \( \mathbb{R}^n \) into two sets of prescribed Gaussian measure, so that the (Gaussian) area of their interface is minimal, is by using two complementing half-planes. This is the Gaussian analogue of the classical Euclidean isoperimetric inequality, and is therefore referred to as the "single-bubble" case.

A natural generalization is to decompose \( \mathbb{R}^n \) into \( q \geq 3 \) sets of prescribed Gaussian measure. It is conjectured that when \( q \leq n+1 \), the configuration whose interface has minimal (Gaussian) area is given by the Voronoi cells of \( q \) equidistant points. For example, for \( q=3 \) (the "double-bubble" conjecture) in the plane (\( n=2 \)), the interface is conjectured to be a "tripod" or "Y" - three rays meeting at a single point in 120 degree angles. For \( q=4 \) (the "triple-bubble" conjecture) in \( \mathbb{R}^3 \), the interface is conjectured to be a tetrahedral cone.

We confirm the Gaussian double-bubble and, more generally, multi-bubble conjectures for all \( 3 \leq q \leq n+1 \). The double-bubble case \( q=3 \) is simpler, and we will explain why.

None of the numerous methods discovered over the years for establishing the classical \( q=2 \) case seem amenable to the \( q \geq 3 \) cases, and our method consists of establishing a Partial Differential Inequality satisfied by the isoperimetric profile. To treat \( q > 3 \), we first prove that locally minimal configurations must have flat interfaces, and thus convex polyhedral cells. Uniqueness of minimizers up to null-sets is also established.

This is joint work with Joe Neeman (UT Austin).